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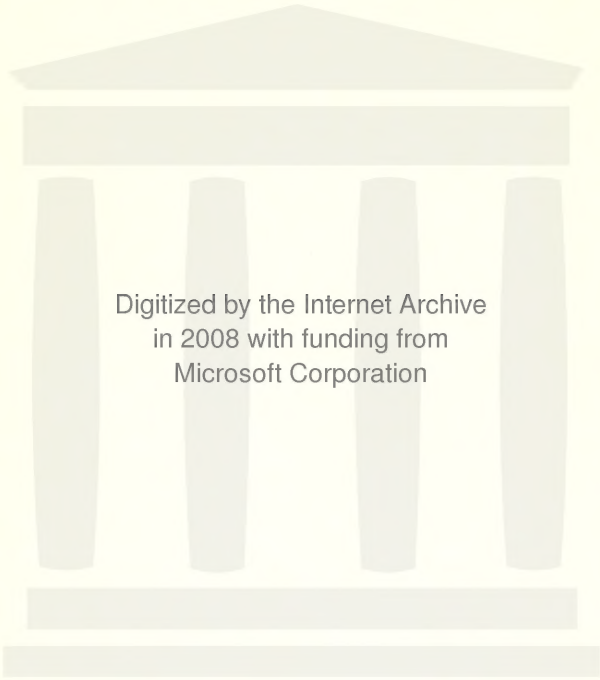
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MODERN DEVELOPMENT OF ARITHMETIC

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THE MODERN DEVELOPMENT OF ARITHMETIC

BY

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NORMAL SCHOOL, REGINA

April 6, 1920.

To the Registrar,
University of Toronto:

We beg to report that the thesis of Mr. John Henry McKechnie entitled "The Modern Development of Arithmetic," together with his discussions of the questions set on the History of Education, the Science of Education, Educational Psychology and Educational Administration, qualify him for the Degree of Doctor of Pedagogy.

H. T. J. COLEMAN.
W. E. MACPHERSON.
W. PAKENHAM.
PETER SANDIFORD.

To the Senate of the University of Toronto,

I hereby certify that the thesis above-mentioned has been accepted for the degree of Doctor of Pedagogy, and that Mr. McKechnie has complied with all the regulations in accordance with the Statute in that behalf.

JAMES BREBNER,
Registrar.

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April 9, 1920.

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The Modern Development of Arithmetic

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The Modern Development of Arithmetic

CHAPTER I.

HISTORY OF DEVELOPMENT.

The modern development of arithmetic begins with Pestalozzi, while its history dates back to the Ionian Greeks. Mathematics is one of the oldest of the sciences. There has probably been no race however simple but has had some idea of number, and likewise no race now existing is without some number sense. Ancient tablets now deciphered indicate that the ancient peoples of Babylon and Egypt had reached a stage of comparative advancement in mathematical learning at least 2,000 years before Christ. One of the more recent and interesting pieces of evidence is the Rhind papyrus in the British Museum. This manuscript was written by one Ahmes, around 1000 B.C., and is thought to be a copy of a treatise written at least 1,000 years earlier. Some even believe that the original dates back much earlier. The manuscript is called "directions for knowing all dark things," and contains problems in arithmetic and geometry, the solutions of which formed part of the secret knowledge of the priests.

Part of the work deals with fractions. In early days these always presented extreme difficulty and their early appearance and prominence in this ancient papyrus makes one think that Ahmes was a mathematician of some rank and that the treatise was intended for mathematicians of equal skill. Another part of the work deals with the fundamental processes. Still another with what might be called algebra. Certainly the ancient Egyptians had some idea of algebraic symbols. The closing portions of the manuscript are taken up with geometry.

In common with the Greeks and all ancient peoples, the Egyptians made little headway with fractions. Ahmes dealt

only with fractions having numerators, one, a plan later adopted by the Greeks. He is able by some rule not divulged, to change other fractions into unit fractions. Other races dealt with fractions having a constant denominator, the Babylonians using 60 as denominator, the Romans 12.

It must be assumed that among early races engaged in commercial pursuits, the art of calculation kept pace with the demands of commerce. The Phœnicians, particularly, were a commercial people and they doubtless handed on to the early Greeks much information on matters of astronomy, navigation and simple arithmetical calculations. Indeed, Greek tradition assigned the development of geometry to the Egyptians and that of science of numbers to the Egyptians or to the Phœnicians.

With the history of Greek civilization begins our definite history of mathematics. The first appearance is in the form of practical arithmetic. The Greeks clearly distinguished two forms. One was called "logistic" and embraced commercial arithmetic; the other was called "number" and with geometry it formed the pure-mathematical branches of philosophy taught in the Greek schools. Commercial arithmetic, or the art of calculating, was passed around among those engaging in trade. The simple processes were done by means of the abacus or swanpan, an instrument in common use among the ancients. The Greek schools of philosophy were not for such tradesmen, and in these schools only the theory of numbers was considered.

Thales appears to have been the founder of the earliest Greek school of mathematics. He was born about 640 B.C., and early became engaged in commercial pursuits. During his frequent visits to Egypt he became interested in mathematics there taught, and he himself became proficient in the astronomy and geometry of his day. Toward the close of his life he opened up schools in his own land, the one at Miletus being particularly famous. The mathematics of his school was largely geometry, while his successors gradually dropped mathematics for philosophy.

Following Thales may be mentioned such teachers as Pythagoras, Archytas, Anaxagoras, Plato, Aristotle, Euclid, Archimedes, Hero, Ptolemy, and many others, with schools

situated mainly at Athens or Alexandria. In all these schools, however, arithmetic, as we know it today, was of concern only as it helped in geometry, trigonometry, astronomy, etc.

It is of interest to note that the first book on the subject of arithmetic was probably that of Nicomachus, 100 A.D. This remained the standard for 1,000 years.

From Greece, mathematics spread eastward to the Arabs and westward to the Romans. The Arabs are also indebted in a limited degree to the Hindus. While little is known of the Hindu mathematics, it is likely they borrowed from the Greeks through commerce, following the latter's occupation of Alexandria and Egypt. The Hindus and the Chinese have always maintained a very ancient existence and a credit for a correspondingly ancient knowledge of all sciences. These pretensions are beginning to be doubted. In the case of the Hindus there seems to have been little science prior to the period of the Aryan invasion. Careful investigations likewise discount many of the pretensions of the Chinese.

However, the Hindus elaborated systems of computations far in advance of any previous nations. They were familiar with the Rule of Three, simple and compound interest, alligation, pipe or fountain questions, series, square and cube root. They were the first to recognize negative and irrational numbers. Negative numbers were conceived of as debts or liabilities, in contradistinction to assets, represented by positive numbers. They also employed the idea of direction. Direction to the left might be represented by a negative number if direction to the right were represented by a positive number. Both these ideas are used by teachers today in explanation of negative numbers in elementary algebra. Algebra was the real field of the Hindus.

The Arabic debt to the Hindus arose through a Hindu scholar, who was called to Bagdad about 773 A.D. It is probable that the Hindu numerals came at the same time. But it is to the Greeks that the Arabs are mostly indebted. Such works as those of Euclid were translated into Arabic at about the time when the Hindu influence was being felt. The first Arab arithmetician made his appearance about 800 A.D., but he was more interested in astronomy than

arithmetic. His name, Alchwarizmi, gradually passed into the form Algoritmi, whence the modern word Algorithm, representing a certain form of computation. This mathematician was quite familiar with Hindu computation.

The Romans were intensely practical and the spirit of their civilization was unfavorable to the development of abstract mathematics. The simple calculations were done by means of the abacus, and abacal reckoning was taught in the schools. Multiplication and division with large numbers gave the Romans much difficulty. Fractions with them were mostly concrete and occurred more frequently in money computations. They formed the major portion of the work done in the schools. As was intimated earlier, duodecimal fractions were quite the popular form used.

The Romans, as early as the first century A.D., developed the method of representing numbers up to 10,000 at least. This method was an extension of the finger symbolism which dates back to very early times. It is not recognized, however, that the Romans made any lasting contributions to the subject of arithmetic. It is from Rome, however, through Boethius, who died in 524, that mediaeval Europe received knowledge of mathematics. Boethius' arithmetic was based largely on that of Nicomachus.

With the death of Boethius little progress was made in mathematics among the Romans. About a century later some activity is noted in Spain, where Isidorus (570–636), wrote an encyclopaedia patterned after some of the Roman works. The quadrivium is discussed, but little insight is given into modes of computation then in use. Little more is known as to the progress of mathematics until the monk, Bede (672–735). His works contain instructions on the calculation of Easter, a mathematical feat in those days. Bede copied the finger symbolism of the Romans.

During the next several hundred years, considerable improvements were made in the abacus, and there finally appeared one with horizontal instead of vertical lines, which was adopted in England, France and Germany.

Through the middle ages arithmetic continued its twofold division, theory of numbers and practical arithmetic, or the art of calculation. In the Universities the theory of

numbers occupied an important place alongside of algebra and geometry. The art of calculating was gradually introduced into the elementary and trade schools, while the Church taught enough to enable the priests to compute the occurrences of religious days.

In the twelfth century, Arabic manuscripts were translated, and Arabic and Hindu methods began to gain a foothold. The abacus was discarded for the principle of local value and the zero of the Arabs, but not without a long struggle. The school opposed to the abacal computation became known as the algoristic, and was assisted materially by the Italian scholar, Leonardo of Piso, also called Fibonaci. It was well into the middle of the fifteenth century before the use of the Hindu numerals was general in England, France or Germany.

Reference should be made to the formation of the Hanseatic League in the thirteenth century. This resulted in schools being established where arithmetic was the chief subject. This was necessary owing to the ever-increasing demands of commerce, for which the arithmetic of the church schools was wholly inadequate. In the League's schools the arithmetic was entirely commercial and unsuited to young children, and consequently not taught to them.

The Renaissance gave an impetus to the teaching of arithmetic, and from this period arithmetic as we know it today dates. It was largely coloured by the arithmetic which the Greeks brought with them from Constantinople, prominence being given to the theory of numbers and geometrical designs. The invention of printing made books possible, and helped to definitely fix Hindu numerals, which fact, of course, made arithmetic more popular.

Between 1550 and 1650 the common symbols of operation, such as we have them, were invented and established. Prior to this, all statements of operations had to be set out at length in writing. About the middle of the eighteenth century decimal fractions became firmly established, and with their establishment business arithmetic was much facilitated. The present methods of multiplication and division also owe their existence to the improvements inspired by the New Learning.

With the advent of the Hindu numerals a great change came about in the method of teaching arithmetic. Prior to that time, object teaching was in general vogue, the abacus and numerical counters being chiefly used. Teachers now saw that the old-time objective teaching was unnecessary and practically abandoned objective work of any kind whatsoever. The result was that arithmetic, which was mechanical before, became even more so, and rules upon rules were manufactured and placed in the texts to be memorized by teacher and pupils. And for over three hundred years this memoriter method held sway, though objected to occasionally by such men as Locke.

The necessity of committing such a mass of definitions, rules, etc., to memory, and the consequent difficulty, led to the invention of rhyming rules which became very popular. Another outcome was an undue attention to form, which resulted in fantastic ways of setting out the mechanical work to the loss, very often, of efficiency.

Arithmetic became a most laborious subject. Fortunately, because of the difficulties involved, it was delayed until the child could read. Numeration was first taught to trillions, then the four fundamental processes in order, and so on. Thus it remained until the close of the eighteenth century, when such men as Trapp, Busse, Rochow, Pestalozzi, and others, began to lighten the burden. Trapp used objects in an endeavour to teach number ideas rather than figures and words. To Busse we owe, in addition, the number pictures, while Rochow tried to make the subject somewhat more interesting. It remained, however, for Pestalozzi to revolutionize primary arithmetic, especially along the line of method. To him must be given the credit for the increased attention which arithmetic aroused in the nineteenth century. The importance which he held for the subject follows partly from his belief that all our knowledge has its origin in the three primary capacities or faculties—sound, form and number.

By his time arithmetic had degenerated into mere ciphering and memorization of facts, rules and principles, and, when taught at all in schools, it made tremendous demands upon the patience and energy of the pupils. Fortunately,

under such methods, it was not taught before the third or fourth year, by which time the child would be able to read and write. Pestalozzi would relegate written arithmetic to a later period, but would begin oral arithmetic immediately upon entry of the child into school. In thus beginning number work early, Pestalozzi maintained he was meeting the legitimate desires of the children, who, he declared, liked numbers as well as they liked letters.

Pestalozzi placed strong emphasis on perception, *i.e.*, the use of objects in developing the number sense. This was but part of his general enunciation that sense impression was the absolute foundation of all knowledge. The application which he made of this principle to number teaching revolutionized the character of the instruction then in vogue; and the excellent results which followed his teaching were the admiration of the many visitors to his schools.

An examination of Pestalozzi's writings and those of his immediate followers would indicate the following as his leading contributions to the subject of arithmetic:

1. He would have pupils "think" in their number work and not merely memorize words, rules, etc.

2. He would have pupils gain their first numerical ideas by means of objective teaching. Then they should pass to representations upon the blackboard. In this way he hoped pupils would grasp number in its abstract character.

3. Number work should begin when the child enters school. The nature of the treatment was to be oral, no written arithmetic to be taken until the number space 1 to 10 had been covered.

4. The processes should not be taught simultaneously.

5. The work was to be characterized by thoroughness.

6. It must be correlated at every stage with language.

7. Pestalozzi gave arithmetic practically the foremost place in school. As training for the mind, he considered instruction in arithmetic unequalled, particularly oral arithmetic.

The influences most felt were perhaps the idea of sense perception as the basis and formal discipline as the aim of instruction.

Pestalozzi's methods were more popular among writers upon educational topics than among teachers. Of the German writers who adopted Pestalozzi's principles, Tillich was perhaps the most outstanding. He corrected his master's omission of due emphasis upon 10 as the base upon which all subsequent numbers are related. Another follower, Turk, stressed the doctrine of mental training, but would delay number until the tenth year of age. This was a radical departure from what the master had advocated.

The over-emphasis upon the formal culture afforded by the teaching of arithmetic, thus giving to the subject a place of supreme importance among school studies, led to a reaction, especially in Germany. Among the more moderate followers of Pestalozzi should be mentioned Kranckes, who adopted a "concentric circle" treatment of numbers throughout the grades, and endeavoured to supply more interesting problems in applied numbers, thus eliminating some of the formalism which attended a strict adherence to Pestalozzian methods.

The methods introduced by Pestalozzi received much currency throughout England, through the writings of the Mayos. Readers of Dickens will remember how these methods are satirized. In the United States the methods became popular through the agency of the Oswego movement and through the efforts of Soldan and Seeley, who translated Grube's works, and thus indirectly helped to spread the reform advocated by the great Swiss teacher. A more direct influence was through Colburn's Arithmetic, the first edition of which appeared in 1821. This arithmetic remained the authority in primary methods for over a generation.

The follower of Pestalozzi who perhaps interests us the most in this study is Grube (1816–1884). His work in arithmetic appeared in 1842, fifteen years after the death of Pestalozzi. Writers attribute very little originality to Grube, and maintain that he was content to copy the good points that occurred to him in the writings of his predecessors. In any event, his work in arithmetic was a thoroughly developed system of number teaching, finding its germ in the unsystematic teachings of Pestalozzi. This book marked

an epoch in number teaching in Germany, and the adoption of the method in Canada and the United States was quite general. In fact, except with enlightened teachers, the Grube method is the one in use today with but slight modifications.

Grube discussed four years' work only, from the sixth year to the tenth. The aim of this period was a thorough knowledge of the fundamental rules and common fractions. Objects are to be used to secure the number relations until the child is able to reproduce these without objects. Grube himself was in favour of a prolonged use of objects, but his followers employed them with more moderation. Grube would teach all the processes simultaneously as against the consecutive order of Pestalozzi. This is considered by some to be Grube's only contribution and innovation on the methods of Pestalozzi, beyond giving extended interpretations to certain principles of the master.

Grube's general plan was the four concentric circles suggested earlier by Kranckes. The first year would embrace the numbers 1–10; the second year, 1–1000; the third year, first half, 1–1000; the third year, second half, 1 to numbers in general. The fourth year was to be devoted to fractions.

In the first year the study began with the number 1. By the addition of another 1 we have 2, etc. Each number is compared with all numbers that precede it, and no new number is to be taken up before the preceding number is thoroughly mastered. Pupils are to be assisted in their grasp of number ideas by the use of blocks manipulated by the teacher. The teacher holds up one block and asks the pupils how many blocks he has in his hands. He gives the block to one to hold. This one passes it back. While this is going on, questions and answers are being carried on, in which great care is exercised that all pupils' statements are in good complete sentences. The teacher would then draw the block on the blackboard and suggest that instead of the block the vertical line | be written to represent the number. Grube's idea here was that the pupil should pass from the object to its picture to the straight mark standing for "one." This was to make the idea of "one" abstract and thought of apart from the block. Next came

the figure 1. In all three, viz., the block, the picture, and the vertical line, there is the idea of "one." Next comes the application. The teacher asks: "What thing do you find but one of in the school room?" Answer: "I find one stove." Question: "What have you one of at home?" Answer: "I have one dog." Other and similar questions followed, applying the idea to everyday realities.

The teacher now goes on to number two. Two blocks are held up and the pupil asked how many. Answer: "You have two blocks." The blocks are now held separately, one in each hand, and pupils are questioned to be sure they see two objects apart. The teacher brings the hands together and asks: "What did I do?" Answer: "You put one and one together." The teacher then places this story on the blackboard:

□ and □ make □ □

getting as much from the pupils as possible. Then, as in the case of the number 1, vertical lines replace the blocks, as:

| and | make | |

The figure 2 is now given. Pupils are practised making the figure 2. The blocks are again held together for the class to see "two" (?). Emphasis is laid on the two-ness, the child forgetting the two *blocks*.

Q. "How many 2's have I?"
A. "You have one 2."
Q. "How many times have I 2?"
A. "You have 2 one times once."
Q. "How many does one 2 make?"
A. "One 2 makes 2."

The development at first will require the teacher to exercise some patience. The two blocks are again held in one hand and the pupils observe. Then one is removed.

Here follow a series of questions, followed by answers in complete sentences. The blackboard is then employed, and so on as before, until all the facts of 2 have been set out, after which they must be thoroughly learned. Next comes the application of the abstract 2 to problems, such

as: Fred had two cents and spent one cent for cherries. How much had he left?

The same method is followed with the other numbers. For instance, the number 6 is studied carefully and the following scheme set up on the blackboard:

0 1	
0 1	$1+1+1+1+1+1=6$
0 1	$6\times1=6 \quad 1\times6=6$
0 1	$6-1-1-1-1-1=1$
0 1	$6\div1=6$
0 1	
002	$2+2+2=6 \quad (2+2=4;\ 4+2=6)$
002	$3\times2=6$
002	$6-2-2=2$
	$6\div2=3$
0003	$3+3=6$
0003	$2\times3=6$
	$6-3=3$
	$6\div3=2$
0000 4	$4+2=6 \quad 2+4=6$
00 2	$1\times4+2=6$
	$6-4=2$
	$6\div4=1$ and remainder 2
00000 5	$5+1=6, \quad 1+5=6$
0 1	$1\times5-1=6$
	$6-5=1$
	$6\div5=1$ and remainder 1

$6=5+1,\ 4+2,\ 3+3,\ 2+4,\ 1+5$
$5=6-1,\ 4+1,\ 3+2,\ 2+3,\ 2+4$
$4=6-2,\ 5-1,\ 3+1,\ 2+2,\ 1+3$
$3=6-3,\ 5-2,\ 4-1,\ 2+1,\ 1+2$
$2=6-4,\ 5-3,\ 4-2,\ 3-1,\ 1+1$
$1=6-5,\ 5-4,\ 4-3,\ 3-2,\ 2-1$
$6=6\times1,\ 3\times2,\ 2\times3$
$3=\frac{1}{2}\times6$
$2=\frac{1}{3}\times6$
$1=6\times6$

Other combinations may be set out by the teacher if she so desires.

As soon as pupil can set out the scheme for any number without the blocks he should do so, but whenever stuck, recourse to the blocks should be made.

After reaching 10, Grube suggests omitting writing the tables out as was done for the other numbers, unless the teacher wishes it for an exercise. By the time 10 is reached, such questions as

$$2\times3+2-1+6-5-3\times2+5 = ?$$

must be done instantly, orally of course. After 10, according to Grube's followers, no objects should be used at all. Some pupils may not need them after, say, four or five.

In the second year the new work embraces the number range from 1–100. Within this range practically the same method is used as in the first year. Each number is to be taken up in turn (after 10), there being 100 distinct steps in the first two years.

The third year is to be divided into parts. During the first half the numbers to 1000 are studied. Each number is no longer isolated as in the first two years. Processes are to be taught, analogy being drawn from smaller numbers when necessary.

Much attention is given to the interpretation of numbers, to separating them into hundreds, tens and units. Splints in bundles are employed to illustrate the steps. The pure hundreds receive the same treatment as the individual numbers did in the first year. For example, take 400:

$$\begin{aligned}
100+100+100+100&=400\\
4\times100&=400\\
400-100-100-100&=100\\
400\div100&=4\\
200+200&=400\\
2\times200&=400\\
400-200&=200\\
400\div200&=2\\
\\
300+100&=400\\
100+300&=400\\
1\times300+100&=400\\
400-200&=200\\
400-300&=100\\
400\div300&=1 \text{ and remainder } 100
\end{aligned}$$

The analysis of numbers was most thorough, practice being given in breaking up numbers thus:

360	
300+ 60	3×100+3×20
180+180	3×120
200+160	10× 36
320+ 40	5× 72
336+ 24	20× 18
etc.	etc.

During the second half of the year, the four fundamental rules in abstract and concrete numbers were covered within unlimited range. The formal processes were evolved here and pupils were called upon to explain at any time.

The work of the fourth year was taken up with fractions and was most exhaustive.

The above review of Grube's method has been brief but perhaps comprehensive enough to show its good and bad points. Professor Seeley has done much to popularize the method in the United States and Canada, and one cannot do better, to place its good features before the readers, than to quote the advantages claimed by Seeley himself. They are as follows:

(1) It recognized the psychological fact that nearly all the knowledge obtained by the child in its earliest years is by means of the senses.

(2) As it makes the first year's work a study of the numbers 1–10, it lays a solid foundation.

(3) The Grube method progresses gradually and naturally according to the ability of the child.

(4) It develops the mental powers evenly and in all directions.

(5) Elementary teaching of number should proceed from observation; or, better, it should proceed from things.

(6) The Grube method makes the teaching of numbers an excellent language lesson.

(7) The child acquires the habit of close observation.

(8) It develops and trains the attention.

(9) It forms the habit of thoroughness in the child.

(10) The Grube method gives pleasure and awakens a love for the study of number.

(11) It makes the child self-active in a proper manner.

(12) The Grube method is a logical one.

It is to some of these so-called advantages that the greatest opposition to the Grube method is now taken, and consideration will be given to them throughout the following pages. A few points will be considered here.

Pestalozzi's insistence on observation in gaining knowledge was most timely in view of the character of the instruction of his day. In enunciating his principle, he had largely in mind the teaching of science where observation of facts is most essential. It is true he carried this into the field of arithmetic. To Grube the appeal to the senses came with a stronger force, and he applied it with all the vigour he possessed to the teaching of number. It was but the reaction of one fully aware of the gross defect of the mere word memorization of his day, and, like all reactionaries, Grube went to the extreme. If by observing an object closely the pupil gains a thorough knowledge of its qualities, why not extend the principle and secure number ideas from the manipulation of objects? He, of course, overlooked the true essence of number, which is a relation and not a quality to be abstracted from things. Sensations cannot give us number, which arise only as a product of the mind in ordering and relating the data furnished by the senses.

It is a matter of common experience that children, before they come to school, or in the first year, employ numbers beyond ten. The dozen is a very common unit to many, while the 25 and 50 cent pieces and the one dollar are familiar to all. In thus limiting the work within the range of 1–10, Grube was not consulting the natural range of the child's experience and interests. The child is hemmed in, as it were, and his memory burdened with combinations leaving no scope at all for the exploitation of child-like motives and for initiative and self-activity in the true

sense. Such work cannot arouse interest, and attention must be forced. Grube recognized the limitations which this limited circle imposed, but he thought the thoroughness of the grounding provided would compensate amply for thus throttling the growth of number sense at its very source.

In beginning with the number 1, then passing to 2, then to 3, etc., after a complete mastery of each, the procedure may appear rigidly logical, but it is doubtful whether the mind of the child works this way. Must the child leave the number 5, for instance, strictly alone until he has mastered 2, 3, 4? Yet many of his out-of-school experiences are associated with the idea of 5. The mind works from within large wholes needing measurement, and not in this piecemeal fashion, by the additions of a fixed unit. Grube's method, in fact, is on a par with the synthetic method of teaching reading, still in vogue in too many places.

Why Grube should depart from Pestalozzi's consecutive order of the four fundamental processes to the simultaneous treatment can only be explained on his unwarranted extension of the sense-perception principle. When studying objects, why not make the study complete and neglect no side of the number idea? And he evidently assumed that all four processes were of equal difficulty, an assumption which any teacher of experience knows to be ill-founded, even apart from psychological considerations. This resulted in a thoroughness altogether foreign to the child mind, and a close application of his method degenerates into a pure routine of memory drills and is utterly deadening. A mere glance at the lay-out required by the Grube method for each number, say, 6, exposes a mass of detail sufficient to swamp any child, however sturdy. Moreover, experimental psychology gives results contrary to this simultaneous treatment. It shows that the smaller the number of habits learned simultaneously, the more rapidly will each of the habits be learned.[1]

[1] "Suggestions of Modern Science concerning Education," by Jennings, Watson, Meyer and Thomas.

The plan of requiring the child to answer in complete sentences is certainly uneconomical and involves a habit which he will not preserve into adult life. Not only Grube but also his master too often overdid the language side, and so warped the development of other lessons to provide opportunities for language training that they became artificial and very often valueless.

Objection might also be raised to the implication of a fixed unit idea throughout. Professor Dewey rightly scores it very severely. We still see evidences of the fixed unit idea in the Unitary method of solving problems, which some teachers force upon their pupils in all its detail, regardless of the fact that the pupils may have solutions more economical and equally logical.

The mistaken idea of number held by Grube accounts for his neglect of the true idea of the fraction. With him a fraction is a part of anything. An apple may be divided into three equal parts. One part, according to Grube, would be called one-third. But a fraction is a number in the same sense that we speak of the number 5, say, and cannot possibly be some *thing*, but a relation of a unit to a multitude. The ratio idea in number comes to clearer recognition in the fractional relation and consequently our idea of 3, say, is not complete until we have mastered one-third. This distorted idea of a fraction which Grube's method evokes accounts for much of the difficulty afterwards encountered in the study of fractions.

CHAPTER II.

VALUES AND AIMS.

Arithmetic is closely related to other branches of mathematics, especially to geometry and algebra, so that in discussing the value of arithmetic as a school study we shall occasionally use the larger term, mathematics, and survey the larger field which this term embraces. In addition to the advantages in thus treating arithmetic in its broader relations, considerable repetition will be avoided in Chapter V, where arithmetic is discussed in its relations to geometry and algebra.

In ages past the aim of arithmetic, when taught at all to the young, was largely utilitarian. This was true in China, India (where it was seldom taught at all), and among the early Mohammedans. Among trading peoples it was essentially so. The Greeks, if we omit the philosophers, had little use for the subject; while at Rome it was regarded as useful for its practical value.

The early philosophers, however, saw something more. To Solon, Plato and others there was in mathematics a valuable training for the mind. This aspect, however, received but little consideration in Western Europe down through the middle ages. By the close of the eighteenth century, however, the practical and disciplinary values of arithmetic and mathematics generally were firmly established.

Other and secondary reasons for teaching arithmetic appeared from time to time. In the sixteenth century it was used by some as a display of knowledge, and one of the famous arithmetic texts of the time employs, quite early in its pages, a number of eleven digits for manipulation.

It was during the same period that Recorde's "The Whetstone of Witte" was written. An examination of the problems contained therein indicates that arithmetic was held in high esteem as a sharpener of wit and as making

one keen and ready in argument. Many of the questions are puzzles, obscure often in their meaning, leaving ample room for argument. Arithmetic was, in fact, encouraged on this account, and became unreasonably difficult.

During the next century appeared several arithmetics on the continent, containing many ludicrous problems, the idea evidently being the amusement of the readers.

In Germany and Italy, for a time, mathematicians belonged to a closed profession, requiring fees for their services and guarding their secrets very zealously.

It has already been mentioned that in the cloister schools interest centred around arithmetic in the calculation of Easter and other church days. The priests also recognized in the subject a value as a training in disputation.

During the last century the utilitarian and culture values fairly well maintained their grounds. One authority,[1] writing early in the present century, sets out the values of mathematics which may be accepted as representing the general attitude at the time. Summarized, they are as follows:

(1) Its utilitarian values, second only to the mother tongue.

(2) As a fundamental type of thought. Mathematics is one of the few characteristic types of human thought; no civilization has ever failed to evolve it, and with essentially the same results.

(3) As a tool for the study of nature.

(4) As exemplifying, especially well, certain important modes of thought.

(5) Other values: generalizing conceptions, combining results, in formation and use of symbolic language, forming habits of neatness and accuracy, cultivating reverence for truth and self-scrutiny, æsthetic side, etc.

Under modes of thought might be mentioned ability to grasp a situation and draw necessary conclusions.

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[1] Young; "Teaching of Mathematics," 1907.

Speaking more particularly of the purposes of the teaching of arithmetic, the same author's summary is as follows:

(1) To teach the child the mathematical type of thought (to understand statements, to observe properties, to make inferences, etc.).

(2) To arouse his interest in the quantitative side of the world around him.

(3) To give accuracy and facility in simple computations.

(4) To impart a working knowledge of a few practical applications of arithmetic.

(5) To prepare the way for further mathematics.

When Professor Young wrote, the conflict surrounding formal discipline had just begun. It is interesting and instructive to read what Professor Smith writes just ten years afterwards, after much of the smoke of battle had blown away.[1]

During the last century the belief in the importance of arithmetic, both on account of its bread-and-butter value and also its culture value, took such root that the subject was raised to a position of pre-eminence among all school subjects. It is only within the last twenty-five or thirty years that doubts have been raised. The Committee of Ten in 1893, and afterwards the Committee of Fifteen, but gave expression to an ever growing feeling when they discounted very materially the practical value so long insisted upon for this branch of mathematics. Writers since then have emphasized their conclusions, and only recently the writer heard one classical scholar declare that the classics were of more real value to the average citizen than mathematics, arithmetic included.

That this time-honoured argument for arithmetic should ever be questioned is due in part to the enriching of the course of studies beyond the three R's and the necessity for curtailment somewhere in order that the new subjects might find a place. Educators, too, were working out

[1] Smith: Address before the Faculty, Teachers College Record for 1917. A brief summary is given in Appendix F.

a new psychology in which "formal discipline" underwent a severe shaking and one of the strongest props for arithmetic teaching was threatened.

As outlined earlier, arithmetic had little place in the schools of Western Europe in mediaeval times, except for what the priests taught. With the increase in commerce, this arithmetic was found entirely inadequate in character, and an impetus for commercial arithmetic was initiated by the Hanseatic League. Naturally all forms of commercial arithmetic, as then employed, found their place in the texts. In the long evolution of the subject, it is only natural that other aims should make their appearance from time to time. Each contributed its type of question to the text book. These combined influences account for the insertion of such topics as equation of payments, rule of three, alligation, foreign systems of money, etc., and many questions of no practical value whatever but inserted for their presumed value as mental gymnastics. Of these, the old "tap" question is an illustration, also the clock questions, and those involving the passing of trains.

As business procedure improved, our texts did not keep pace in the elimination of antiquated topics. And the teacher, dependent upon the text, taught all such topics regardless of their inapplicability to modern business practice. The same may be said of the other types of questions. There was no haste to get rid of them, even when the doctrine of formal discipline had undergone reconstruction.

The Committee of Ten, mentioned above, dealt with secondary mathematics, but its report is instructive to the teacher of elementary arithmetic. "The Conference was, from the beginning of its deliberations, unanimously of the opinion that a radical change in the teaching of arithmetic was necessary. . . . The Conference recommends that the course in arithmetic be at the same time abridged and enriched; abridged by omitting entirely those subjects which perplex and exhaust the pupil without affording any really valuable mental discipline, and enriched by a greater number of exercises in simple calculation and in the solution of concrete problems.

"Among the subjects which should be curtailed, or entirely omitted, are compound proportion, cube root, abstract mensuration, obsolete denominate quantities and the greater part of commercial arithmetic. Percentage should be reduced to the needs of actual life. In such subjects as profit and loss, bank discount, and simple and compound interest, examples not easily made intelligible to the pupil should be omitted. Such complications as result from fractional periods of time in compound interest are useless and undesirable. The metric system should be taught in applications to actual measurements to be executed by the pupil himself; the measures and weights being actually shown to and handled by the pupil. This system finds its proper application in the course which the Conference recommends in concrete geometry. . . ."

The report was quoted at some length on account of it being the pioneer effort in the reform of arithmetic. Since this report was written, several writers have emphasized the need of curtailment and elimination of much of the material usually found in our arithmetic texts. The suggestions below will be considered quite conservative by most of the readers:

(1) Questions in the four fundamental rules employing unreasonably large numbers, as the following: $897650463286 7 \times 974864069$, are not uncommon.

(2) Much of the work in fractions, especially addition and subtraction with fractions whose denominators are greater than 16. There should be more facility in the manipulation of fractions commonly used, such as those with denominators of 2, 3, 4, 5, 6, 8, 12 and 15.

(3) Greatest Common Divisor. The reduction of a ratio by G.C.D. is seldom ever done. It is reduced to a decimal.

(4) Least Common Multiple, except the very elements. Finding the L.C.M. by inspection should be emphasized.

(5) Cube root.

(6) Compound proportion.

(7) Equation of payments, alligation, partnership and most of partial payments.

(8) Interest except direct simple interest and the mere elements of compound interest. A little practice in use of tables might be helpful.

(9) Indirect commission and brokerage such as: A man sent his agent $2,000 with instructions to deduct his commission of $2\frac{1}{2}\%$ and invest the balance in wheat. How much did the agent invest?

(10) Much of the present work in insurance, exchange and stocks.

(11) All tables and units of measures now obsolete, like troy and apothecaries' weight.

(12) Metric system. Let it be taught if needed in the elementary science.

(13) Longitude and time, except as necessary in lessons in geography, on standard time.

Many of these topics have disappeared from some of the arithmetics. Where they have not, the teacher should make a careful elimination along the lines suggested.

Pupils leaving school should be:

(1) Accurate and reasonably rapid in the four fundamental processes.

(2) Familiar with decimals and accurate in use of the decimal point.

(3) Familiar with and able to use the common units of weights and measures.

(4) Able to solve problems embracing common business practice.

(5) Able to estimate fairly approximately answers to problems.

(6) Able to apply the knowledge gained in class-room to simple everyday matters outside.

Coming now to the culture value derived from a study of arithmetic, we should note that in the field of formal discipline mathematics long held the supreme place. No training was regarded as quite equal to the training from mathematics. The doctrine of formal discipline had

developed out of the educational system of the middle ages and was in harmony with psychological tenets of the time. Until the last few decades nothing arose to cause doubts either as to the doctrine or the faculty psychology which supported it. It was firmly believed that mental power, however gained, is applicable to any department of human activity. Common experience seemed to bear out this doctrine, and even when doubts were raised, some of the experimental work, notably that of Meumann, lent confidence to such a doctrine.

Professor Smith, writing in 1909,[1] says: ". . . . What shall we say? That arithmetic has no mental discipline that other subjects do not give? No one really feels this, in spite of the fact that the exact nature of the discipline is hard to formulate. Everyone is conscious that he got something out of the study, aside from calculation and business application, that has made him stronger, and the few really scientific investigations that have been made as to the effect of mathematical study bear out this intuitive feeling. . . ."

We cannot here even survey the controversy which has raged around the doctrine of formal discipline. It has been attacked vigorously by such men as Thorndike, O'Shea and others, but it has not been without its defenders. Some of the more aggressive statements had later to be modified. Professor Thorndike advocates rather the doctrine of identical elements, while Professor Judd substitutes that of generalized experience.

Professor McMurry[2] is very mild in his condemnation: "The doctrine of mental discipline, of logical exercises through arithmetic, being grounded in a once-prevailing psychology, is still held by many. But in the last few years the changes in psychological theory and in the course of study have been so great as to shake even this stronghold of the old arithmetic. The doctrine of formal discipline has been largely undermined by the leading schools

[1] Teachers College Record; January, 1909. Professor Smith speaks with much more confidence on the subject of formal discipline in the May issue, 1917.

[2] McMurry: "Special Method in Arithmetic," p. 8.

of modern psychology, both in America and Europe. The old idea of separate faculties of the mind developed by different studies, for example, the logical faculty by arithmetic, has now little recognition among psychologists. The doctrine of apperception has become in the main a substitute for the old notion of formal mental discipline. . . ."

We must admit that the teacher of mathematics is very much at sea on the question of formal discipline. It is very comforting to read what Miss Rogers says in Teachers College Record for September, 1916. Miss Rogers sums up the matter succinctly when she maintains that at the present time no psychologist of repute denies that transfer is possible. It is not a question of transfer but degree of transfer and how it takes place. Even where the transfer is very small, yet a very small spread of training may be of great educational value provided it covers a sufficiently wide area.

The writer also points out that most of the experimental work showing small amounts of transfer have been based upon adult studies. With children, doubtless a much better showing would be made, especially if performed under actual class-room conditions.

Several have investigated how transfer takes place, among them being Thorndike and Ruger. The latter found four factors operative in transfer of training—ideals, attitudes, concepts of method and high level of efficiency -which suggest certain practical injunctions:

(1) Proper attitudes should be cultivated in the pupil.

(2) Attention should be focused on the art of learning and on methods of procedure in the solving of problems so that the pupils should be stimulated to analyze the situation, to formulate hypotheses, to criticise and evaluate each suggestion, to be systematic in selecting and rejecting these and in verifying them. Further, each step should be generalized as a method, so that there should be deliberate control of assumptions.

(3) Attention should be directed to related ideas, in order that as many as possible may be recalled or discovered.

(4) Motivation should be secured and attention should be kept at a high level.

By such means the experience gained in mathematics will tend to be generalized and made available in other fields.

Whether mental discipline is real or not, few will be found to advocate that modern business problems should give place to those applicable to antique business practices. No quarrel can be raised over the pruning recommended in previous pages.

When the value of arithmetic as mental training was being questioned and materially discounted, emphasis was shifted to its business utility. The complaints and criticisms of business men against the character of arithmetic taught in the schools and the quality of the scholars turned out had been incessant. Now, schools began to lend an ear to the business man, and business practices were studied as the basis of method in the class-room. The advent of new subjects of study was good reason for curtailing the content and teaching just those things needed in business life.

With the ever-growing conception of education as a social process, thoughtful teachers are beginning to ask in what way may arithmetic, as one of the important school subjects, help to make the boy and girl socially efficient. This broad social aim for school studies is opposed to any tendency to use the school for specific vocational training, and arithmetic under this conception should be taught so as to develop a truly broad socialized culture.

In this light, arithmetic ceases to be a mere matter of skill in calculation or a tool subject, but, if rightly taught, contributes social insight just as geography, history and other subjects may do. Thus the business man is not the school's proper critic, but rather the sociologist, and school does not fit for the needs of the business man any more than those of any other profession. This view results in bringing the material of school arithmetic into closer relation to the experience of the children and the needs of the community, while a closer correlation is established between arithmetic and the other subjects. In all grades the pupils' out-of-door life on the quantitative side is transferred to

the school and made the basis for the work in arithmetic. In this way the work has point and motive not otherwise obtained. Greater use is made in the lower classes of number games, the play instinct, etc. Objective work ceases to be artificial. Before entering school, the pupils have acquired limited number ideas in the natural way. A continuation of this should be the method of the primary teacher. Opportunities for the incidental teaching and correlation with other school activities are made use of but not forced. In some schools there is no prescribed course in arithmetic into which the pupils must fit themselves. The various games, school activities, manual training and domestic science classes, and the various other school subjects, afford ample material for the class-work in arithmetic.

In the senior grades the various institutions of the community are studied, their place in the life of the community and the problems incident thereto. The problems are therefore real in the best sense of the word. Arithmetic becomes broader than mere figuring. "Arithmetic without a pencil" has a new meaning. Pupils do not get their first ideas of a bank, interest, discount, etc., from reading the text-which is often meagre and misleading in its explanations. A visit to the bank is made, or pupils are asked to make certain observations prior to a certain date when "Banking" is to be the subject of discussion. Then follow several periods of discussion, without a pencil, on the purpose of the bank, its place in the life of the people. The problems will grow out of these discussions in a natural way, and pupils will have the setting of them. They know the situations out of which the problem evolves, and have a concrete background from which to reason.

Only in this way will arithmetic cease to be so abstract and formal, so weighted down with obsolete matter which the pupil has no way of comprehending. It will be something vital to him. Real motives will be provided and interest will permeate it all, resulting in arithmetic interpreting for him, quantitatively, the larger social life for which he is being prepared.

CHAPTER III.

THE PSYCHOLOGY OF NUMBER.

It is important to note that no race has yet been discovered devoid of number ideas. These ideas may be very primitive, like the one, two, many, of the native tribes of the Malay Peninsula. The "one" may be the mere "this," as against "two," the mere "that." As the needs of the race multiply, the number ideas enlarge and a corresponding number vocabulary is evolved. These expanding needs, the need for more exact measurement in the process of adjustment to environment, we shall see, account for all number growth in the race; and to this degree at least the development of the child parallels the development of the race. This need supplies the "motive" underlying the development of numerical ideas, and should never be lost sight of by teachers.

But however primitive the number ideas may be, no language is without the suggestion of number, and one is almost justified in the conclusion that the primitive conception of number is fundamental in thought. Certain recent experiments[1] with kindergarten children prove that a succession is created in young children long before there is any conscious idea of number. This consciousness of succession or series-idea which is created is rhythmical in character and contains the potentiality of numbering which must soon find expression in the activity of counting.

These changes in succession occur to the child of the savage as well as to the child of more refined parentage. They are the result of the varying impressions from all the senses. In case of the child born into high civilization the materials for sense impressions will be more numerous and varied, and we should not be surprised if this resulting consciousness of succession is more striking.

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[1] Phillips: "Pedagogical Seminary V."

The fact that a race has a paucity of number names does not necessarily imply a paucity of number ideas. Number names, as with language in general, never keeps pace with thought. Number capacity lags behind its linguistic mode of expression, and the savage, by means of gestures, or by use of stones, sticks, etc., is able to express number ideas far in advance of his numeral language. The original inhabitants of Victoria, Australia, had no numerals beyond two, and yet they counted and recorded the phases of the moon.

The fact that even the lowest of the human races are not without their number sense has led to speculation as to whether or not this capacity extends downwards and is possessed by some of the higher animals. It is pointed out how animals are aware of the loss or removal of even one of their young, though some half dozen or more may remain. Leroy's crow[1] is, of course, notorious. It is related how a crow had been pestering a man's fruit, but had always managed to evade the gun. The owner of the orchard conceived a plan to outwit the crow. Two men approached the watch-house; one entered; the other passed on. The crow would not approach the fruit. Next day three men were sent; but the crow remained obdurate. Next day the number of men was increased to four, and the same tactics employed. No result. Not until five or six were sent did the crow lose its apparent consciousness of number and feel safe to again attack the fruit. The conclusion drawn was that the crow could count to four.

One is often surprised at the intelligence of trained horses. Only today, at the Exhibition, we were entertained on Johnny Jones' Midway by a very clever horse. At the request of the trainer, this horse would indicate, by pawing the floor, the number of days in the week, the number of working days, and many other "stunts," to the wonder and amazement of all. Still more marvellous are the feats in mathematics performed by the Elberfeld horses,[2] which have long astonished the hundreds of visitors to their stalls.

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[1] Reported by Conant: "The Number Concept."
[2] Hobhouse: "Mind in Evolution," Appendix.

The savage child makes slow headway from one to two, to three, etc. The concept of two will be clear, though he may have no name for it, as soon as he is able to distinguish himself from some other person, or, better, one object from another object. The similarity, in some languages, between the names for one and for two, indicate that little advancement has been made beyond a "this" and a "that." The count is now one, two, many; and many a race stops here. So many, that we are led to conclude that it is a halting point of greater or lesser degree with all races and with the present-day child. It is interesting to note here that the names for one and two do not in all languages have relation to the fingers or to the names of fingers, which suggests that with some peoples at least the fingers have not yet been called into requisition. We see connection more often in the names for three, four, etc.

Where tribes are found who count no further than two, three or four, a duplication of these is often used for higher numbers, namely, two twos, two threes, two fours, etc., as the case may be. We have then the elements of the binary, ternary or the quaternary scales. While such primitive number systems are found in Australia and South America particularly, they are not confined to these two continents. Almost similar conditions exist, or did exist, in Africa, Asia and North America. The Bushmen of South Africa used a name for three which means simply many. The Veddahs of Ceylon have but two numerals. The natives of Lower California cannot count beyond five, while few of them understand two fives. Many of the Eskimos have number systems little better than those of the African and Australian native races.

It should be remembered here that the degree of perfection of a numeral system is not an infallible index of intelligence. Often savages acquire remarkable skill in reckoning by long practice in trading and bargaining. In general, however, growth of number sense increases with civilization, a statement which is almost axiomatic. In considering the limited extent of numerical ideas among tribes of low civilization, we are apt to regard our own abilities as too superior. If one stops to reflect for a moment, he

will realize how really limited our own sense is in this matter. How many have a clear conception of what 1,000 means, let alone 1,000,000? Few people can estimate even approximately the number present in a crowd or other gathering.

In this same connection it is well to recall that few of us have any definite mental presentation of individual numbers, proceeding by differences of one, beyond 100 or perhaps 144 (the 12 times 12 of our school days). We break larger numbers up into convenient units; for example, 500 is grasped as five 100's, or ten 50's, etc., and not as one more than 499. We clearly make use of some denary scale. With numbers such as 39, 75, 96, however, all below 100, most people enjoy definite comprehensions and conceive of each as a single number, quite apart from the three 10's and 9 feature. In other words, we really neglect the base until 100 or so is reached. It is just possible that we over-insist in relating numbers to the base or radix 10 in our schools.

Without a doubt the first visible signs of the number sense is in counting. Behind this counting will be found all degrees of number development. What and where are the very beginnings of this more or less crude number idea, first giving expression in the mechanical one, two, three, etc.? Is there a number instinct? The investigations of Phillips mentioned a moment ago help us to answer the questions raised.

It is familiar to all how children very early can repeat a number of sounds; for example, taps on the table, dad's barks in imitation of the dog, etc. Quite small children of the farm, living where party lines furnish telephone connection among the neighborhood, will play away quite unconscious of the frequent ringing of the 'phone. But the moment the rings occur, which mother has been in the habit of answering, a child will drop everything and become all excitement until the call has been answered, when he will reclaim his toys and play away, again unconscious of the incessant ringing, until his own call strikes his ear. Considerations such as these, supplemented by the data furnished by his investigations, led Phillips to conclude that

long before a child has any conscious number ideas, he is observed to follow a succession of stimuli which creates in him an idea of succession, or a series idea. There is now a consciousness of succession, and it assumes a rhythmical form. It is characteristic of the human mind to group ordered sense impressions into rhythmic multiples, and no one, not even the adult, can get away from it. All sense experiences combine in creating this idea of succession, which is at bottom the number series. "Number seems to signify primarily the strokes of our attention in discriminating things. . . . We amuse ourselves by the counting of mere strokes, to form rhythms, and these we compare and name. Little by little in our minds the number series is formed. . . ."[1]

This series idea is abstract, though it requires sense impressions to create it. Children become filled to overflowing with this idea of ordered and rhythmical succession, hence the passion for counting as soon as number names are supplied. Children have been waiting for just such a sort of expression. Previous to this, their various activities, such as throwing down definite numbers of toys, wiggling the toes, nodding the head, etc., have been the forms of expression. But the articulatory response to this inner subjective feeling or need has the strongest appeal. That this series is abstract and subjective in character is evidenced in the fact that at first children will count with no reference to objects at all, and if asked to count objects, will often let the words run ahead of the objects. That later, objects are found useful as a check, or perhaps restful, was pointed out in the opening chapter. This abstract and "inner" character of the series idea is another evidence that "number is not got from things, it is put into them."[2] It is well worth remembering at this point that these number names are not applied to objects, but are merely a motor symbolism for the inner series, which the child has been developing since birth, through the impressions continually knocking for admittance at the doors of the various senses. The

[1] James: "Psychology," Vol. II, p. 653.
[2] McLellan and Dewey: "Psychology of Number."

richer the child's experiences, that is, the more numerous and variegated, appealing to all and not merely to one sense, the earlier will this series be formulated and the more urgent becomes the necessity of supplying him with some set of names, preferably our one, two, three, in order to relieve this ever-growing and pent-up consciousness of succession. Otherwise the child must seek an outlet somewhere, and the number forms so common to many are the result.

The next step with the child is to direct this counting activity into definite channels, in order that his objective needs and experiences may be more accurately defined. In other words, we try to give objective measurement to this inner and subjective series, "an adjustment between the inner and the outer." This numbering activity, which is purely mental, must be *put into* the child's objective world, in order that he will have a correct idea of it, on the quantitative side. As a rule, the child comes to school with this numbering activity fairly well developed, through his application of it to his toys, playmates, games and activities generally. This suggests to us a method of further development upon the child's arrival at school.

We have seen, then, that quite early in the child's life is created number potentiality as the result of the impressions of ordered phenomena upon his consciousness. This must find scope for expression, and this is provided out of the necessity of the child conquering his environment quantitatively. McLellan and Dewey have shown how the idea of limit is the primary idea in all quantity. "If everything that ministers to human wants could be had by everybody, just when wanted, we should never have to concern ourselves about quantity. . . . It is because we have to put forth effort, because we have to take trouble to get things, that they are limited for us, and that it becomes worth while to determine their limits, to find out the quantity of anything with which human energy has to do."[1] From the viewpoint we have adopted, were it not for this idea of limit there would be nothing of a worth-

[1] McLellan and Dewey: "Psychology of Number," p. 36.

while nature to satisfy this inner impulse and consequently no number development.

This check to the satisfaction of our needs or our activities necessitates an adjustment, a balance as it were, between means and end, or, in others words, accurate measurement. With the child or the savage, whose ends are immediate and easily satisfied, there is not the same effort, not the same requirement, not the same accuracy in adjusting means and end, not the same exact measurements necessary, as with the civilized adult, and consequently the child's or the savage's number ideas remain more or less indefinite. Only as the need for closer measurement is felt and becomes conscious does the number concept stand out in clearness. This adjusting of means and end is the motive behind the measuring activity, and, in thus making the vague whole definite, does the true idea of number show itself.

We have seen how fundamental is measurement in any definite notion of quantity. This implies something, more or less vague, and worthy of being measured, and also some unit of measurement. The product of the mind in performing this activity, the times idea, is the very essence of number. Questions of "How many?" and "How much?" are involved, and answered by means of the mind's activity in relating and counting. Real motive is thus provided for the exercise of that inner number potentially emphasized a few moments ago. In the child's own way, it may be a question of "How many marbles have I?" "How many more sticks do I need for this house?" "How much plasticine will it take for Tiny Bear's bowl?" "How much candy will five cents buy?" and so on. And from the child's simple needs there is a gradual increase in complexity to the needs of the mature adult, but the problem is always the "How many?" the "How much?" There is some vague whole to be made definite, to be measured, and in the last analysis this means a counting of the number of times the unit is repeated. So we repeat again that the idea of number is traced to measurement and measurement back to adjustment of activity.

If this is true, number is a psychical *process*, not a sense fact. Number cannot originate in sensation. Here

Pestalozzi, and particularly his followers, went wrong. Having hit upon a good principle involving sense perception, they were satisfied with a more or less passive attitude on the part of the child, in the presence of objective material, and expected definite number ideas to develop. But, as Professor Judd has said somewhere, no amount of illustrative material will give rise to ideas of number. Huxley was partially true, at least, when he asserted that mathematical science is a study which owes nothing to observation, nothing to experience, nothing to induction, nothing to causality. Merely to look at objects or to listen to sounds will not give the child any idea of their number. Number is not a quality to be abstracted from objects. It is not a property grasped by any of the senses. Only as the mind "orders the objects—that is, compares and relates them in a certain way,"[1] do ideas of number enter into consciousness. Before these definite number ideas are conceived there is a consciousness of quantity, but this is vague, awaiting illumination. This may be the permanent condition of the higher animals, referred to earlier. The savage often gets little further; his wants are few. Once a need is felt, we break the bounds of the indefinite, unmeasured quantity, and bring it by measurement into definite consciousness.

This operation of making a vague whole definite gives number. The presentation of objects must be accompanied by a relating activity upon the part of the child—a discrimination and generalization—both mental processes. Discrimination implies the recognition of objects as units, that is, as distinct individuals;[2] generalization is the grouping together the like objects or units into a sum, having neglected all qualities, save just enough to have the object stand out as *one* (abstraction).

It has been mentioned earlier that Pestalozzi and Grube were wrong fundamentally, in beginning to teach number ideas with a single thing. There must be a group of things upon which the mind may work, having felt the

[1] McLellan and Dewey: "Psychology of Number," p. 61.
[2] Watson: "Outlines of Philosophy," p. 82.

need, by the two mental activities of discrimination and generalization. The natural beginnings must be some whole needing measurement, and not as with Grube, who would start with one thing and then go to two things, etc. This view also opposes the custom of limiting the child's number field in the first year to the numbers 1 to 10.

Summing up, we would keep in mind that number involves three factors: a quantity to be measured, a unit of measurement, the times the unit is counted off or repeated in comprehending the quantity. This idea of times, we pointed out, is number, and, strictly speaking, is a ratio.

Confusion arises when we regard this unit as always being a single unmeasured thing. It may be a unity of measured parts, and not a "fixed" unit, as Grube would seem to imply. We have, for instance, a group of oranges, say, 144. If our unit of measurement is the dozen, the answer to the question how many dozen will be 12. We have an answer perfectly intelligible. If we count in threes, we shall find the number of such to be 48, etc. We may afterwards, of course, convert the number into one where the single one is the unit, our other unit being itself a quantity of measured parts. This idea of a fixed unit may be overcome by practice in the early stages, counting by twos, threes, etc.

Having consideration for the paucity of number ideas in some races, and the difficulty experienced in expanding these ideas,[1] we are inclined to agree with Munsterberg,[2] to an extent at least, in his opinion that the development of the consciousness of number is a slow and a late one in the child's mind, and arises much later than the space apperceptions. Because a child counts, he reminds us, is no indication that he has many clear number concepts. It is doubtful, he says, whether a real consciousness of number relations beyond four or five is reached before school age. This same opinion is expressed by Mr. Rusk.[3] The sensory

[1] This phase of the subject is well treated in Conant: "The Number Concept," and readers are referred to it.

[2] Munsterberg: "Psychology and the Teacher."

[3] Rusk: "Experimental Education."

activity of the child, his perception of space relations, his linguistic powers, and, in part, his technical skill, are highly developed before he arrives at a comprehension of number, according to Mr. Rusk, who, continuing, points out that we are often misled, because a child recognizes the loss of an article, into the belief that he has number ideas. "As the child, in his daily life, in his play, etc., has every inducement to count and yet does not avail himself of his opportunities, we must conclude that number cannot depend on mental processes easy to him."

This is a view which I cannot endorse entirely. Observations would tend to establish the conclusion that counting comes very early, and is of delight to every child. Granted that children have difficulty in the early stages of acquiring number, this difficulty is over-emphasized by teachers. Not only is the child burdened with number to the exclusion of other interests, but the over-insistence upon objective treatment deadens any enthusiasm the child would naturally have for arithmetic.

In connection with the difficulty experienced by very young children in acquiring number concepts, considerable experimental work has been done to test memory for numbers. The results would seem to establish that all types of special memories show periodic variations in their development. They develop at various rates with different children, and differ with boys and girls. What is of special interest at present is that it has been found that development of the memory for abstract conceptions proceeds parallel with that for numbers, whereas the memories for objects and sounds are distinct from and precede those for words and numbers.[1]

In addition to the interest this arouses as to the time to begin formal arithmetic, it would also tend to support the contention that in numbering the mind organizes the data of sense in a time relation, and that no analysis of the idea of space can ever bring forth the idea of number. "Objects, in so far as they are conditioned by the relation of space, are co-existent. But they can be numbered only

[1] Rusk: "Experimental Education."

in so far as they can be thought of as discreet, *i.e.*, as successive."[1]

As we have seen in the preceding pages, the question whether number is related primarily to time or to space is not new. It has been fruitful of debate ever since the issue was first raised by Kant. Sir William Hamilton is typical of the time-extremists; Herbart of those who declare that number is no more related to time than to many other concepts. Lange was moderate, but favoured the relation of number to space.

The question is, of course, important, because there is a tendency today to return to the days, largely pre-Pestalozzian, when exercises in counting was the method of beginning the teaching of number, and there may be a danger of going to the extreme.

Opposed to counting is the view that the idea of number arises from the simultaneous visual perception of objects, and advocates of this view would begin with number pictures, rightly basing their claims, historically, upon Pestalozzi. Many psychologists see both factors, the spatial and temporal; and hence to them the exclusive use of either method is incomplete. Where the spatially based method is employed to the neglect of the temporal element, the child has to supply the omission, and *vice versa*. Any method which does not take both features into consideration is simply postponing difficulty. Howell[2] has the right idea toward the question. He says: "The child enters school with the counting psychosis well established (at least, in its first stage); it is the part of wisdom not to ignore it, but wisely to take advantage of it. But, for the *economical* development of number concepts, the use of number pictures to follow or accompany the seeing, counting and handling of objects is indicated. Through practice with them the ability to grasp groups is strengthened, and percept, image and symbolization all are brought about with economy of time and effort."

The question is still further involved by the variability of endowment types among pupils. The visile will be

[1] Rosenkranz and Lang: "Groundwork of Number," p. 5.
[2] Howell: "Foundational Study in the Pedagogy of Arithmetic."

at some disadvantage if number pictures are avoided; while the audile is favoured by successive rather than simultaneous presentation of stimuli. The question of number pictures will be further discussed in Chapter IV.

We have already remarked that Pestalozzi, Grube and their followers were in error when they affirmed, or at least acted on the assumption, that number is an object of sense perception. Number, we have tried to show, is the result of reflection, of an activity of our minds. On the other hand, we derive our idea of number from the presence of the world external to the mind. Even the concepts of the pure mathematician had their origin in physical experience analyzed and clarified by the reflective activities of the human mind. Conant[1] shows that as number originates among savages it is entirely concrete. The limitations of their intellect preclude anything else. Once the number concept is grasped, real progress is made. The use of numbers in the long run is economical and effective just because numbers are abstract and serve to shorten processes of thought.

McLellan and Dewey, after commenting unfavourably upon the method of "symbols" of the pre-Pestalozzian days and upon the method of "things" which followed, attempting to "abstract" number from things following observation, say: "It is, then, almost as absurd to attempt to teach numerical ideas and processes *without* things and to teach them simply *by* things. Numerical ideas can be normally acquired, and numerical operations fully mastered only by arrangements of things—that is, by certain acts of mental construction, which are aided, of course, by acts of physical construction; it is not the mere perception of things which gives us the idea, but the *employing of the things in a constructive* way. . . . In reality it (number) arises from constructive (psychical) activity, from the actual use of certain things in reaching a certain end. This method of constructive-use unites in itself the principles of both abstract reasoning and of definite sense observation. . . ."[2]

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[1] Conant: "The Number Concept."
[2] McLellan and Dewey: "Psychology of Number."

CHAPTER IV.

SOME ASPECTS OF NUMBER TEACHING.

In this and the succeeding chapter certain aspects of arithmetic teaching will be considered. Some of these have become prominent through controversy and have been the subjects of experimental work. Others are phases which at present need more intelligent treatment; while others again are worthy of more prominence than they receive at the hands of the average teacher.

It is not the intention of the writer to go into minute details of method. As experience grows, one becomes more convinced that it is futile to lay down definite steps of procedure. One is in hearty accord with Professor De Garmo when he admonishes teachers that method must not be regarded as an overlord dominating all the doings of the teacher, but rather as a guiding friend, pointing out the shortest path to a desired goal. While there are a few general principles applicable, with suitable modifications, to all subjects, there is no universal method for any subject, and the teacher is more effective when he applies the principles of method in accordance with his own individuality. Professor De Garmo might have safely added, the individuality of the pupil.

There certainly is no royal road to success in the teaching of arithmetic. In the past we have had our various so-called methods—the Grube, the spiral, the ratio, etc.—and advocates supported their respective favourites with apparently logical principles; but each method was found, in time, to be narrow and to exhibit some psychological weakness. Any method, if employed with consummate skill and energy, will be fruitful of results, because the child is so responsive. Too often, however, we are treated to a teaching device which some enthusiast has mistaken for a method. Teachers, however, in avoiding extremes of

method, should nevertheless feel free to experiment within some limits and not be wedded too much to conservative methods.

Above all, the teacher must remember that it is not the text he is teaching, nor the subject, but the child. Ever since Rousseau's time there has been an increasing reverence and sympathy for childhood. This has resulted not only in improved externals of education, such as good school plants, compulsory laws of attendance, health regulations, etc., but a scientific study of the child, giving us a rich store of psychological information upon which to base our instruction. No longer, at least in enlightened communities, do we impose adult views upon the child, but he is made the centre, the pivot as it were, and the conditioning element in all methods of instruction.

This scientific knowledge, based on child study, has furnished us with broad principles of method, applicable to any child irrespective of the nature of the medium of instruction, whether it be arithmetic, history, geography, or any other of the school subjects. These principles may be found in any standard work on general method.[1] Within these general principles there is a wide range, allowing for the individuality of teacher, the variability of pupils, the local conditions, and the other hundred and one elements entering into successful development of a live boy or girl. There is no one best method to be used with rigidity at all times and under all circumstances and conditions.

In mental development, education simply brings out of the mind what is already in it potentially.[2] Education cannot create; it can only develop. So that the pupils of any grade represent as many variations in mathematical abilities as there are individual pupils. "No head for mathematics" may be an absolute fact. We have always recognized the variations in performance, but have acted on the assumption that these need not be, and by forced methods have attempted to level up the class to the attainments of the best; in other words, to produce a uniform

[1] For instance McMurry: "Elements of General Method."
[2] Sandiford: "Mental and Physical Life of School Children."

product. Variability in its true significance must be fully appreciated by the teaching body before the child will receive just treatment. The statement of Professor Young[1] that mathematics calls for special talent and calls forth special mental activities is an opinion shared by recent educational thinkers.

In the outset the teacher should have a reasonably comprehensive view of the aim of education; reasonably, because no one has yet grasped its full significance. The aim in arithmetic should conform to this general aim. The best thought of today emphasizes the function of the school in training for social service, and we tried to show in a previous chapter how instruction in arithmetic fell short in its contribution to this training if it did not give the pupils a social insight into man's total environment. To do this, the work in arithmetic ought to be derived much more largely from the pupils' own needs and should utilize their spirit of enquiry more than is done at present. School material is valuable only in so far as it furnishes opportunities for full expression and expansion of powers inherent in each child and crying out for development. With such material no artificial stimulus is necessary. The pupil is interested, he is self-active, and as the end to be reached becomes more purposeful, and more depth is given to the child's thinking, the greater is his effort; but interest is present all the time.

WHEN TO BEGIN ARITHMETIC.

The question when formal arithmetic should be taught for the first time has been under discussion for some time. The Committee of Fifteen gave it as its opinion that arithmetic should not begin until the second school year and should be completed at the end of the sixth. While authorities are fairly well agreed that "formal" arithmetic has no place in the first year of school at least, this opinion has not found general expression in our courses of study. Perhaps much depends on our interpretation of the word "formal."

[1] Young: "Teaching of Mathematics."

With Pestalozzi, we have seen, number work began when children entered school and continued throughout their course, Pestalozzi maintaining that young children took to numbers as kindly as they did to letters. His belief, too, in the importance of numbers as one of the sources of knowledge and its value for mental training would be reasons for its early incorporation in the school curriculum.

This early incorporation was entirely opposed to the custom of the schools of his time, but the success which met his efforts and the enthusiasm of his followers resulted in fairly general acceptance of his scheme, in the English-speaking world at least. While there have been protests from time to time against this Pestalozzian plan, there was little combined effort before the report above referred to. On this side of the water, in fact, we have outdone even Pestalozzi, and, following Grube, demanded a thoroughness in Grade I, within the number space 1 to 10, which is quite unpsychological.

Foremost in the attack against the early adoption of arithmetic may be mentioned such men as Hall, Mosso and Patrick. The latter contends that the child at this early age is unable to deal in abstractions which underlie a real comprehension of number, and attributes the "grotesque" number forms, which he condemns severely, to the desperate attempts of the child to give some kind of bodily shape to the abstractions which are forced upon him.

Professor Burnham,[1] viewing the question in the light of hygiene of instruction, contends that the subject should not be taught before the age of 8 or 10. If this is observed, there will be no necessity for the artificial devices and methods employed and which so often lead to arrests or interference of association later on.

Rusk quotes Munsterberg with approval in urging the postponement of the formal study of arithmetic. The number idea is relatively late in making its appearance. Because the young child can glibly count to ten or beyond is no sign, in his opinion, that he has grasped number relations. In fact, experience shows that many of the child's powers,

[1] In the "Cyclopedia of Education," Volume I.

notably his linguistic powers, his perception of spatial relations and his general sensory activity are in advance of his understanding of number. And the ordinary child of six does well if he comprehends the numbers up to three or four.

My own experience leads me to express the opinion that the average child has a much larger comprehension of number than Mr. Rusk suggests. Invariably teachers find this, and have considerable difficulty keeping within the limits, 1 to 10, prescribed by the course of study for the first year, even in schools where no undue time is given to the subject. Even granted that these observations represent what is generally true, it is no argument for an over-emphasis on formal work during the first year.

In his suggested "Course of Study in Arithmetic," McMurry[1] makes no provision for regular number work for the first year. He finds that it is better that the children gather number experiences incidentally from home and school activities. The time spent in drilling on the number combinations may better be employed in broadening the child's experiences in nature and human affairs. Pupils who have not been burdened with this regular and systematic work will do much better work the second year and easily overtake the others.

On the other hand, Smith[2] is of the opinion that the idea of postponing the formal study of number until the second year has not yet impressed itself very seriously upon the minds of educators. He admits that the child might spend the twenty minutes a day, devoted to numbers, more profitably, but that this is not likely until such times as play has become more systematized, thus ensuring that this time shall be devoted to exercise in the open air under the guidance of a skilled teacher. Furthermore, to omit arithmetic from the curriculum of the first year, relying upon the teacher to teach it "incidentally"[3] would mean very perfunctory work in the majority of cases. Added to this,

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[1] Appendix B.

[2] Smith: "The Teaching of Arithmetic."

[3] Suzzallo: "The Teaching of Primary Arithmetic is very doubtful of this incidental treatment."

it is poor pedagogy to concentrate the work into a few years. Rather should it be spread over practically the whole school career. Only in this way can number be properly impressed upon the child mind.

Most teachers will steer a course between any extremes. In our rich courses of study today, with their provisions for games, stories, constructive work of all kinds, the child will meet many number experiences, making the subject of much interest to him, and there need be no great amount of the "formal" side. The truth is that if the life of the class-room is to bear some relation to out-of-door experience, arithmetic, which is but the quantitative side of such experience, must have a real place during every school year. Points of contact between school and home and environment will suggest themselves to all thoughtful teachers.

OBJECTIVE TEACHING.

Since the adoption of the present system of numerals and prior to the middle of the nineteenth century, objective teaching was seldom if ever employed. Before our present numerals became general in use, objective teaching was the common and necessary method. Additions and the other simple operations could not be performed without the aid of numerical counters or the abacus. Arithmetic, we have seen, in those days was most laborious under the clumsy system then in vogue. With the advent of the new numerals all such mechanical devices were thrown to the winds. The old use to which they had been put was not now necessary, and no need at all could be seen for their continuance. Soon arithmetic became more rule of thumb than ever. Number facts and rules for computation were copied from the teacher's text and memorized by the pupils. Little progress could be made with such methods of instruction. Anyone who mastered the Rule of Three was considered a real mathematician, and was much sought after to solve business problems. This result followed inevitably from the necessity of writing down the mass of material to be memorized, oral arithmetic became an unknown activity and arithmetic became impossible for a child before he could

read. To some advocates of the postponement of arithmetic, this may be regarded as not an unmixed evil.

To Pestalozzi, as we have seen, this method of beginning with a mastery of rules and principles was unpsychological. He was not perhaps original in his opposition to the prevailing practice, but nevertheless he is given the credit for revolutionizing methods of teaching primary number work. He believed that every subject should be analyzed into its simplest elements and then gradually increased in complexity, his maxim being "From the simple to the complex." As a further point of method, he maintained that observation or sense perception is the basis of instruction. The number work with actual objects was with him followed by work with charts, upon which were dots and lines, each being considered a unit.

As mentioned before, Pestalozzi was quite generous as to what objects might be used. He had no objections to the fingers. The teacher himself, however, performed whatever manipulation of the objects was necessary. The pupils observed and answered questions asked by the teacher, no written work being allowed at this age.

The enthusiastic reception and adoption of Pestalozzi's methods have already been described. In nearly all cases, however, the spirit was lost, and his followers went to extremes never dreamed of by the master. Arithmetic and objective teaching was no exception, and only under recent psychological treatment is a more rational use of objective teaching being worked out.

It was intimated above that Pestalozzi's actual work with objects held the pupils in a more or less passive attitude. Pestalozzi was a firm believer in self-activity of the child, but it was just another instance of the failure of his theory and practice to harmonize. We owe not a little to Froebel's insistence on self-activity, a doctrine which he reinforced continually by actual class-room practice. Undoubtedly other subjects responded more rapidly to this doctrine than did arithmetic. We see in the present tendency, altogether too rare, to utilize games, and to relate the work to the home and environmental experience of the child, a recognition of the soundness of the Froebelian doctrine.

It may be rightly contended that Pestalozzi's partial indifference to the selection of material for objective treatment resulted in the work becoming artificial and unrelated. His own enthusiasm made up for much of this defect, and this can also be said of many of his followers and of many earnest teachers today. Observation was with Pestalozzi made rather the end in itself. It was frequently overlooked that observation should proceed under the guidance and direction of some definite problem to which it is relevant. Teachers have very glibly repeated the maxim "From the concrete to the abstract," and acted as if mere presence and manipulation of objects induced ideas of numbers.

Professor Dewey,[1] commenting upon the concrete *vs.* the abstract, says in part: "Since the concrete denotes thinking applied to activities for the sake of dealing effectively with the difficulties that present themselves practically, 'beginning with the concrete' signifies that we should at the outset make much of *doing;* especially make much in occupations that are not of a routine and mechanical kind and hence require intelligent selection and adaption of means and materials. We do not 'follow the order of nature' when we multiply mere sensations or accumulate physical objects. Instruction in number is not concrete merely because splints or beans or dots are employed, while whenever the use and bearing of number relations are clearly perceived the number idea is concrete even if figures alone are used. Just what sort of symbol it is best to use at a given time —whether blocks or lines or figures— is entirely a matter of adjustment to the given case. If physical things used in teaching number or geography or anything else do not leave the mind illuminated with recognition of a meaning beyond themselves, the instruction that uses them is as abstract as that which doles out ready-made definitions and rules; for it distracts attention from ideas to mere physical excitations.

". . . . Things and sensations develop the child, indeed, but only because he *uses* them in mastering his body and in the scheme of his activities. Appropriate

[1] "How We Think," p. 139.

continuous occupations or activities involve the use of natural materials, tools, modes of energy, and do it in a way that compels thinking as to what they mean, how they are related to one another and to the realization of ends; while the mere isolated presentation of things remains barren and dead. . . ."

In commenting upon the new movement toward improvement in teaching mathematics, Professor Young[1] has somewhat the same idea in mind. The term "laboratory method" may very well be applied to the character of the new movement. The dominating thought is a better understanding of the child's mind, an attempt to be psychological rather than logical, making the centre of gravity of instruction move in the child's needs and capacities rather than in those of the adult. Whether or not the material and manner of instruction are suitable will be answered by the degree of interest aroused in the child. The child is not interested in the utility of mathematics, nor is he aware of any particular culture value. He is concerned only with "doing," with exerting his powers to their fullest extent. The teacher, school, etc., are of use to him so far as they are sources from which emanate things for him to do, and he is pleased and likes the doing when he finds he can do them.

The same writer goes on to warn teachers against the lavish employment of concrete phenomena lest the child's standard of interest is again usurped by the adult's, and we are no better off than under the memoriter methods of a century ago. All this objective work is in vain unless it has some direct connection with the child's own experience and his own activity.[2]

The laboratory method is of course not peculiar to the field of mathematics. It is the method of all science. It simply insists that the experimental origin of mathematics be taken advantage of to the fullest extent. It is the inductive method, and its employment has done much to correct

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[1] "Teaching of Mathematics."

[2] G. Stanley Hall: "Adolescence," Vol. II, ch. xvi, emphasizes the value of abstraction and the dangers of dwelling too long on the concrete.

the wasteful manipulation of objects and to point to their more purposeful use. But even here many teachers exceed what is necessary to establish certain mathematical facts. Long after the child is convinced that 3 and 2 are 5, the teacher persists in exemplifying the truth, until real boredom sets in. Teachers often labour under the misconception that every beginner must be placed in the presence of blocks, toothpicks, etc., regardless of the fact that he may have had, and likely did have, a fairly rich number experience prior to his entry of school, and that this experience was secured in the natural way in his daily occupations. When Tony, braver than the others, could stand it no longer and exclaimed to his teacher, "Do you tinks I'm a tam fool?" he but spoke what many a lad carries in his heart.

Of late years objective teaching has broadened considerably in its meaning. The use of any concrete experiences, of sense perceptions, to give a clearer idea, a better concept, may be termed objective teaching. Thus it is no longer limited to the mere use of objects in the class-room. An excursion to the local bank, to a building under construction, or to an elevator, for the purpose of observation whereupon to base future problems, may be considered objective teaching. The use of pictures, diagrams, graphs, number games, playing store, etc., are only a few of the varied forms which objective teaching may take.

This broadening conception of the meaning of objective work is seen in another quarter. It has usually been assumed that the need for objective treatment decreased with age. While considerable presentation of objective material was considered good in Grade I, it was a sign of weakness if employed in the later grades. The attitude now is that the newness of the topic is the determining factor. The pupil's experience, rather than his age, will be the proper criterion. This view awaits experimental investigation, but in the meantime many teachers, under the conviction that the view is sound, treat such topics as square root, fractions, and various topics in mensuration and commercial arithmetic, in their introductory phases, objectively.

One form of objective teaching, the school excursion, is only occasionally made use of and yet it should be the most

fruitful. The benefit of travel was recognized by such early educators as Montaigne and Comenius. Their idea was travel after school-education was complete. Travel as a corporate part of school came later. Basedow introduced it into his Philanthropinum at Dessau. School excursions are now the common thing among German elementary schools and are also popular with some English schools. On this side of the water they are not as frequent as they should be. When used they are for the benefit more particularly of such classes as geography, nature study and natural history.

It is only recently that the possibilities in such excursions for arithmetic have become apparent. As we get away from the old idea of arithmetic as a "tool" subject, and see in it possibilities for social insight as great as are possessed by many other subjects, we realize the need for studying at first hand the various institutions of our community and thus securing a concrete background for the various problems which grow naturally out of the interrelations of such institutions and the life of the people whom they serve.

To illustrate: The study of banking, interest, discount, etc., should be preceded by a visit to the bank, either by the class as a whole, or by individuals prior to the day upon which the discussion is to take place. In this country, after a succession of wheat failures, the farmers' wives with the help of the children are turning their attention to the creamery. A visit to the local creamery would give the boys and girls a new light upon the industry in which they are vitally interested and with which so many problems in arithmetic are associated. Instances might be multiplied of advantageous excursions in the field of arithmetic.

Another form which objective teaching has taken of late is "dramatization," a common instance being "playing store." Teachers find this of real help when teaching such topics as our coins and many phases of commercial arithmetic.

Diagrams, graphic illustrations and graphs may be used to advantage, and are fairly familiar to most teachers. The use of the graph is becoming very common in newspapers, books and magazines, and pupils should be familiar with their

interpretations. The record of the school teams may very well be set out by means of the graph; so may the variations in temperatures, etc. The use of the graph is almost unlimited.

Within the last decade or two, considerable use has been made of arithmetical apparatus for elementary mathematical work, especially in mensuration. In some of the countries on the continent, use is made of angle measures and scale drawing for field measurements. Such apparatus as the mirror for measuring heights, the mirror-angle for running perpendiculars and computing distances, the pocket compass, the protractor, graduated staffs, etc., are all used and found very helpful in the older classes. These have not been adopted to any extent in our schools, but their possibilities are evident.

NUMBER GAMES.

Present-day school teachers have been slow to utilize the play instinct in primary number work. Without a doubt, the value of play has been recognized from time immemorial. Primitive peoples today recognize its importance in a way. Ancient peoples did not fail to see something of value for education in play and games. Heaven, to the Egyptians, was a place for music, dancing and games. Plato was the exponent of play in the education of children and his advice to mothers on nursery play is suggestive even now.

We may put it down as a fact that the Greeks were the first great advocates of play in education. They had their games of the nursery, the gymnastic exercises of the school and the agonistic exercises and social games for mature life. The rattle, ball, hoop, top, toy carts, doll, etc., were familiar to the Greek children. Plato advised against too many toys, believing that originality and invention were hindered where everything was ready made. Out of doors, such games as Odd and Even, Hunt the Slipper, Hide and Seek, were the everyday games of the boys and girls of that time.

When at seven years of age the Greek boy went to the palestra, his mornings were largely given over to spirited play. At sixteen he advanced to the gymnasium, where exercises were still an important part of the school day. Many writers credit the wonderful temperament of the Greeks, and their sense of the aesthetic, to the physical part of Greek education and see a coincidence between the abandonment of Greek games and the decline of Greek sculpture.

In the long interval between the Early Greeks and Froebel, play was not generally recognized as having any educational value. When it was not regarded as harmful it was tolerated in the absence of more profitable tangible employment. Occasionally one arose like Rabelais (1483–1553), who saw something in play. Many trace his influence on Locke and Rousseau. In his book, "Gargantua," the manners and education of the sixteenth century are satirized and the prevailing scholastic formalism of his time is violently opposed. He proposed to teach by play and have his pupils learn even mathematics through recreation and amusement!

Erasmus, Comenius, the Jesuits, Fenelon and Locke all saw educational value in play, games and physical exercises. Perhaps the attempt of Basedow is the most outstanding. But it is from Pestalozzi and Froebel that modern educators have caught the real spirit of play in the education of the young. Both owe much to Rousseau, who first seriously impressed upon educators the necessity of child study.

While both Pestalozzi and Froebel advocated the doctrine of self-activity, Froebel was the precursor of modern child study, and enunciated a theory of recapitulation, which required that the child must be allowed to be himself at each succeeding stage. To him play was one of the child's highest modes of self-expression. He says: "We should not consider play as a frivolous thing. . . . On the contrary, it is a thing of profound significance. . . . By means of play the child expands in joy as the flower expands when it proceeds from the bud; for joy is the soul of all the actions of that age." Froebel's kindergarten and the method

by which he converted play into systematic teaching are too familiar to need more than mention here.

The significance of play in education led many to formulate theories regarding it, all assuming that it is instinctive in origin. The overflow-of-energy theory advanced by Spencer and Schiller was also basic with Froebel, with this difference: that he would emphasize the mental rather than the physical aspect. Lazarus, in Germany, proposed the recreation theory. Play is for relaxation and recreation of exhausted powers. Dr. Stanley Hall advocated an atavistic theory, finding in play the motor habits and spirit of the past of the race persisting in the present. McDougall sees in play the impulse of rivalry, to get the better of the opponent, in a friendly way of course. The good and bad points of these theories have been discussed by others;[1] it is only necessary to say here that all have some good features. It remained for Groos, however, to discover that play was actually a preparation for the business of later life—a theory which has been almost revolutionary from the standpoint of educational theory and practice. Play immediately assumes a place of high importance in the learning process.

The puppy growling and barking at some meaningless object, or engaging in mock encounters with its mother and brothers; the kitten springing after the moving ball; the girl working over her doll, are engaged in activities which are preparatory to similar and more serious pursuits at maturity. Thus play is nature's schooling, and should be supplemented but not interfered with by man's schooling.

Professor Horne[2] shows how play and work stand in contrast. Play is for its own sake as far as the subject is concerned, and is always pleasant. In work there is some ulterior end to be attained, which may not in itself be pleasant. It is in play that the individual is able to preserve himself and his freedom. "It keeps the springs of personal being ever fresh and flowing." In work the individual subjects himself to the demands of the environment. Some enthusiasts, professing to be followers of

[1] Sandiford: "Mental and Physical Life of School Children."
[2] Horne: "Philosophy of Education."

Herbart and his doctrine of interest, would sugar-coat all instruction and school activities, so as to reduce effort to a minimum. In school, according to Professor Horne, the place of play is fundamental beside work, affording the necessary reaction from work while preserving the individuality of the pupil; but to do everything playfully is to remain always a child. Teachers would be well advised, then, not to lose sight of effort while imparting instruction. "As children outgrow the kindergarten, the work impulse must appear increasingly prominent. Play remains, but not as the sole spring of action."

One is inclined to wonder, however, if it is not possible to approach work in the spirit of play. It is questionable whether children put forth more real effort in work than in play. Are not the sources of joy in work, enumerated by Eliot, after all the chief sources of pleasure in play? They are: the pleasure of exertion; achievement, particularly competitive achievement; co-operation, involving harmony and rhythm; exercise of judgment, intelligence and skill; encountering risks, danger, making adventure.

Perhaps enough has been written to show how important play and games are in the educational field. Arithmetical or number games are but one phase of the larger field of educational play and games. In order that school play may be employed to the best advantage, a knowledge of child development is essential. The teacher should know the various stages into which child life may be conveniently divided for special study, keeping in mind that there are no real divisions, although the divisions usually made conform to empirical observations and have some scientific basis. It is not thought wise, within the limits of this thesis, to enter into an analysis of these divisions. Such an analysis may be found in any good text in psychology.

In the appendix[2] will be found a partial list of suitable games with brief descriptions. These are taken from Teachers College Record, 1912, p. 392.

This field of objective teaching is comparatively new, and further study in this field will prove helpful. A

[1] Teachers College Record, January, 1918.
[2] Appendix C.

recent experiment[1] was undertaken to see to what extent games involving number are of interest to children; what games afford possibilities in the matter of points of departure for formal work in arithmetic; to what extent a knowledge of number can be gained through these games. While results were more or less indefinite, the value of number games was evident. Many instances were afforded for motivated work in arithmetic, the need for a knowledge of number being very keenly felt at times. Classes where number games were used exclusively apparently showed higher improvement than classes where games and formal work were undertaken. The investigator reports a very important by-product of the games, namely, development of the spirit of social service.

Number rhymes, while not games in the strict sense of the word, may well find a place here. The child has a delight in metre. We remarked elsewhere how our minds are attuned to rhythmical responses, and that no generation has been devoid of this characteristic. And so with peoples; all, from the most primitive, tend to versify in the philosophy, history, religion and emotional life. It is only natural, and hence to be expected, that mathematics should respond to this racial habit. India is an example where mathematics was set down to metre, like their ancient religious classics. We have an obvious relic of this primitive law in the delight which children find in the rhythm of counting away and beyond any immediate needs.

Our nursery rhyme,

> One, two, buckle my shoe;
> Three, four, bolt the door;
> Five, six, pick up sticks;
> Seven, eight, lay them straight;
> Nine, ten, a big fat hen,

is taken from very ancient times; while our "Thirty days hath September . . ." dates back probably to the twelfth century at least.

Evidence points to the fact that, as far back as number was taught, rhymes existed, and when texts were printed,

[1] Teachers College Record, January, 1918.

following the invention of printing, these contained such devices as aids in learning the subject and for the amusement they afforded. An interesting children's book of the early nineteenth century has this rhyme to aid in learning the multiplication table for facts of three:

3 times 1 are 3,
My darling, come to me.
3 times 2 are 6,
The man has brought some bricks.
3 times 3 are 9,
This boy's a friend of mine.
3 times 4 are 12,
I find no rhyme but delve.
3 times 5 are 15,
Lead the donkey on the green.

With the advent of the numerals in the middle ages, verses for their dissemination became very numerous. Methods of learning and teaching arithmetic were entirely memoriter following the introduction of these numerals, and all the rules and principles were put into verse, and likewise all instructions how to learn tables, etc. This tendency to versify remained with English writers for several generations. At times the problems appeared in verse—a practice common in the time of Euclid. An example from an English text is the following:

When first the Marriage-knot was tied
Between my Wife and me,
My age did hers as far exceed
As three times three does three;
But when ten years, and half ten years,
We Man and Wife had been,
Her age came up as near to mine,
As eight is to sixteen.
Now tell me, I pray,
What were our Ages on the Wedding Day?

That these must have remained popular for several generations I can bear testimony, for this rhyme was often rhymed to me by my mother when I was a boy.

It would seem that at the present time little or no use is made of versifying. Improved methods of teaching initiated by Pestalozzi undoubtedly accounts for this absence of the use of rhyme. One wonders, however, whether or not a little could not be used to advantage with young children.

The mathematical puzzles and tricks of several centuries ago have entirely disappeared. The reason cannot be found in pupils' dislike for them, for one has only to point to the interest youngsters take in the puzzle department of the children's corner in the newspaper. They lose interest, of course, if they are unable to solve these puzzles.

It is interesting to note that some of the so-called puzzle questions are being used as tests for intelligence.[1] For instance, the old familiar one: "A mother sent her boy to the river to get seven pints of water. She gave him a 3-pint vessel and a 5-pint vessel. Show me how the boy can measure exactly 7 pints without guessing at the amount."

Another source would be the curious cases which are quite common. We are told that the continental countries make constant use of these. A few may be illustrated here:

A. Have the pupils perform the following:
   1. Write any number consisting of 3 digits.
   2. Write the number obtained by reversing the order of the digits in this number.
   3. Subtract the smaller from the larger.
   4. Write the difference obtained and write the new number formed by reversing the order of the digits in this difference.
   5. Add this new number to the difference found in 3.

   All the pupils' answers will be 1089.

B. Multiply 12345679 by 9, then by 6:

$$\begin{array}{r} 12345679 \\ \times\quad 9 \\ \hline 111{,}111{,}111 \\ \times\quad 6 \\ \hline 666{,}666{,}666 \end{array}$$

[1] Terman: "The Measurement of Intelligence."

C. Set this out, having the pupils perform the multiplications:

$3\times37=111$
$6\times37=222$
$9\times37=333$
$12\times37=444$
$15\times37=555$
$18\times37=666$ etc.

D.

$2\times142{,}857=285{,}714$
$3\times142{,}857=428{,}571$
$4\times142{,}857=571{,}423$
$5\times142{,}857=714{,}285$
$6\times142{,}857=857{,}142$

There is no introduction of new digits in the product.

E. Tables such as the following prove interesting:

$1\times9+2=11$
$12\times9+3=111$
$123\times9+4=1111$
$1234\times9+5=11111$
$12345\times9+6=111111$
$123456\times9+7=1111111$
$1234567\times9+8=11111111$
$12345678\times9+9=111111111$
$123456789\times9+10=1111111111$

NUMBER PICTURES.

The existence of number pictures dates back to the time of Pythagoras at least. The speculations of Pythagoreans were all related to number. Pythagoras himself made many discoveries concerning numbers, and employed considerable symbolism, using dots arranged in symmetrical patterns. Each number had its own particular significance.

The first systematic employment of number pictures for number teaching belongs to Von Busse, towards the close of the eighteenth century. Pestalozzi, who followed him, gave a more scientific turn to their use, and may be considered the inventor of the scientific use of number pictures. Number ideas were secured from the manipulation of objects; but to relate back the concept to the perception material, after the first numerical ideas were gained, *Strichtabelle* or stroke-tables were employed. These were the forerunner of the present-day number picture.

It was pointed out in Chapter III that the upholders of the spatial elements in number advocate the exclusive use of number pictures as against counting to develop number ideas, whereas supporters of the temporal aspect would limit the practice to counting. The matter has been investigated by several, some employing observation and introspective methods, others experimental. Meumann contends that both the spatial and temporal elements are involved. Lay, by experimental methods, reaches the same general conclusion. Teachers would do well to work on the assumption that only by a combination can the real essence of number representation and processes be understood by the child.

On the assumption that number pictures have a rightful place, the question which remained was to determine the best form they should take. Pestalozzi employed lines largely, but circles or points have been proven superior. Recent investigations would also indicate that in the spatial arrangement of things such matters as distance apart, size and the direction their arrangement takes, play an important part, while vividness or brightness is also a factor. These are points, however, which require further investigation. It has also been suggested that advantage should be taken of the sense of touch.

The forms below have been the object of study:[1]

1. 1 11 111 1111 11111 111111 1111111 11111111 111111111 etc.

2. 0 00 000 0000 00000 000000 0000000 00000000 000000000 etc.
   0 0 00 00 000 000 0000 0000 00000

3. 0 0 00 00 000 000 0000 0000

4. 0 0 00 00 00 0 00 0 00 00 00 00 00 00 0
   0 0 00 00 00 0 00 0 00 00 00 00

5. 0 0 00 00 000 000 000 000 000 000
   0 0 00 00 000 000 000 000
   0 00 000 etc.

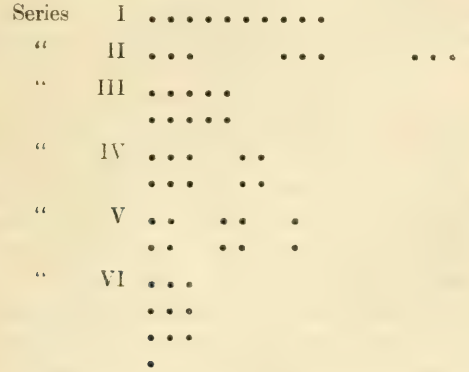
6. 0 00 00 00 00 00 0 0 0 0 0 0 0
   0 00 0 00 0 00
   0 00 00 00 0 0 0 0 0 0 0
   0 0 0 0 0 0 etc.

[1] Rusk: "Introduction to Experimental Education" and Freeman in "The Elementary School Teacher, March," 1912.

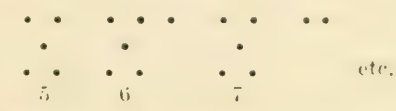
The first is the form commonly used by Pestalozzi. Lay found that the one-row arrangement, as in 1 and 2, is unfavourable, and that the arbitrary grouping of 6 is unsatisfactory. He found 3 and 4 the best arrangement, with slight advantages in favour of 4, the so-called quadratic arrangement.

Walsemann found 3 and 4 many times preferable to 1. Number 3 was superior to 5. As regards the normal *versus* the quadratic arrangements, his experiments favoured the normal.

Freeman used a tachistoscope (a short-exposure apparatus) to find which form was most easily apprehended. Some of the groupings used were as follows:



It was found that children are less correct than adults in estimating the number of points, underestimating very often; young children less correct than older children. Complex forms are less easily apprehended than simple. The single horizontal row, as . . . . . , is better than a vertical. When the rows were doubled there appeared to be no preference. For children, groupings in five were best, as



For adults, groupings of four were found better.

It will be noted that the investigators are not in entire agreement, and the subject will need to be investigated more before the best arrangement is decided upon.

NUMBER FORMS.

The question of number forms has assumed some importance of late owing to recent investigations and the consequent controversy as to their good or bad effects in early mastering numbers.

Many persons think of numerals in some form of visual imagery. If the idea of six, say, occurs to them, the figure of 6 rises before their mental eye in a written or printed form. These same persons will doubtless have "forms" for days of the week, months of the year, kings of England, etc. Colours often assume spatial representations, as also sounds.

While number forms have perhaps always existed, Galton was the first to give us any definite information on this interesting subject. Since then, several investigators have given the matter considerable study.[1] Galton had no number forms, and only discovered them in another accidentally. Persons who have them treat the matter as quite an ordinary thing, and are surprised to learn that there are many who have no such forms. They certainly are not common among adults. The writer has none. Galton estimated that they were present in 1 out of every 30 male adults and 1 out of every 15 female adults. An investigation among my students, all of whom were over 18 years of age, gave higher proportions. Phillips found they existed in 7 per cent. of males and 8 per cent. of females.

All subjects questioned state that these forms date as far back as the memory extends. These forms have a definite direction with each person, and the numbers have a definite distance at which they appear. If such persons were asked to describe their number imagery at intervals of several years, they would place such forms in the same shapes and positions. All such forms have some points in

[1] Notably Patrick and Flourney in 1893. Calkins and Phillips later

common, yet each has his own peculiar and often grotesque shape, and arithmetical operations are done by jumping from place to place on this shape with astounding rapidity.

Mr. Galton[1] reprints several letters from correspondents who describe their peculiar number forms. Here is a typical letter: "From the very first I have seen numerals up to nearly 200 range themselves always in a particular manner, and in thinking of a number it always takes its place in the figure. The more attention I give to the properties of numbers and their interpretations the less I am troubled with this clumsy framework for them, but it is indelible in my mind's eye even when for a long time less consciously so. The higher numbers are to me quite abstract and unconnected with a shape. This rough, untidy production is the best I can do towards representing what I see (sketch not shown). There was a little difficulty in the performance, because it is only by catching one's self at unawares, so to speak, that one is quite sure that what one sees is not affected by temporary imagination. . . . About 200 I lose all framework."

The subject was investigated among my classes and diagrams secured. Those students who had number forms thought it quite natural, and were surprised that they were in the minority. Their attention had never been called to these forms by any other person, but they were present with them as long as they could remember, and were likewise made constant use of. In describing these, Miss N. wrote: "It seems rather difficult to describe the number forms. The numbers seem to start directly in front of me and to go up to 10 in a gradual ascent straight before me; 11 and 12 are even with 10, and then there is a sharp turn to the right and a sharp curve downward to 20. From there, there is a gradual slope to 100; the numbers such as 40, 50, etc., are a little more prominent. After 100 the number form is so confused in my mind with the places of the numbers that I cannot tell where it goes. Thus, when I think of 200 it seems to have some place on the form, yet it also calls to mind the taking of three places to the

[1] Galton: "Inquiries into Human Faculty." 1883.

left. The drawing[1] does not seem to express it very well, as the whole thing is straight in front of me and tending only very slightly to the right."

The same student had forms for the months, days of the week, and for sounds, but none for colours. Of sounds, she says: "Some voices seem to me as round, some rectangular, etc. I remember once making my sister laugh by describing a singer's voice as 'too loose around the edges.'"

Miss S. wrote: "I use this in addition and subtraction. I always see the numbers at regular intervals. If I think of 34, for example, I always think of it as so many spaces above 30. If I think of 36, it is so many spaces below 40. I have a similar curve over 100. I never remember having the numbers arranged like this by the teacher."

This student had forms for the months, days of the week and letters of the alphabet.

Mr. B. wrote: "It always seems to me that the numbers 1, 2, 3, 4, 5 are in the dark, and beginning with 6 they appear in the light. The number 10 appears larger than the rest, as though it were a marking post. The numbers 15, 20, 25, etc., up to 50, are a little larger than the ordinary, but not as large as 10. The number 50 is a large one, however, and some way marks a division. . . . My associated ideas of the fact that 70 to 79 is a long line, and 80 to 89 is short, is that of playing Hide-and-Seek. In counting from 10 to 100, it takes longer to say the numbers from 70 to 79 and less time to say from 80 to 89. . . ."

Mr. B. had also interesting forms for the months and days.

Miss B. wrote: "I have noted a decided break at 11, others less pronounced seem to come between 14 and 15, 24 and 25, 49 and 50. The heavy lines are to indicate vividness. Some numbers are larger than others, all even numbers being larger than odd, but 10, 12, 15, 24, 36, 60, 75 and 100 are larger than the rest. The number 16 seems to overshadow or be almost on top of 17, the same is true of 24 over 25, 42 over 40 and 81 over 80."

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[1] Drawings are not shown here.

Galton noted that the higher numbers rarely fill so large a space as the smaller numbers. He was of the opinion that the diminution of space occupied by them followed a geometrical law, but he was unable to discover it. He also noted that the forms do not curve in the same direction with all persons, the right being more prominent. Nor do they always lie in the same place. These observations were verified by the evidence of our students.

Of all the forms, number seems to be the oldest. Galton thought they came into existence when the child is learning to count and is used by him as a natural mnemonic diagram. At first, spoken words, "one," "two," etc., are referred to it, afterwards the visual symbol figures, which then become permanently fixed. Probably a great many children employ such diagram, but when they become too faint to be of service are gradually neglected and eventually forgotten. With those who continue them, they increase in vividness, and in some adults are therefore quite pronounced.

None of my students could trace any relationship between the number forms and the clock. Galton attributes the origin of some of the number forms to the number material used in the class-room, such as dominoes, abacus, etc., the configuration of the country, hills, dales, etc., surrounding the child's home. But the greatest influence, he thinks, follows from our nomenclature, which is far from uniform in the early stages. The difficulty pupils must be labouring under is evidenced in the hitches and twists evident in the diagram. He thinks—an opinion borne out by others—that in many families this curious tendency is hereditary, and found instances where the number forms in some families were alike. No evidence on this point could be secured from my classes,[1] but I was struck by the similarity between the forms of some of the students and those given by Mr. Galton.

[1] In a later class of students, a sister of a former student possessed a number form. Comparison of the forms of brother and sister showed no real similarities. The young lady was not aware that her brother made use of any form whatever. She was surprised that so few members of her class had no forms at all. She had always regarded her own as a matter of course and no one had ever called her attention to it.

Investigations would seem to bear out Galton's opinion that number forms of some kind or other are almost universal in children at some period in their lives. One writer gives a genetic explanation of their origin. He recalls how early counting is a motor response to a series idea, rhythmical in character, which is formed in the child long before he has any number words at his command.[1] This inner feeling or idea is so urgent for expression that it must burst out in some form or other, and most often takes its concrete symbolism in geometric form.

No investigation that I am aware of has been undertaken to ascertain by means of parallel classes, say, whether these forms are beneficial or not. My students, when questioned, did not know. None would admit being considered particularly bright in arithmetic. Some positively "hated" mathematics, and declared that arithmetic had always been their bad subject. Some authorities are of the opinion that good number forms should be cultivated as aids to memory and reproduction and thus to elementary operations. One[2] suggests adopting a uniform number form and hanging it up to be constantly before the children.

On the other hand, number forms are condemned as evidence of forcing formal number work upon the child too early.[3] This writer sees in the development of such habits interference of association and the germs of pathological neuroses. Too early is the mind filled with quantitative ideas to the exclusion of causal relations.

Howell pooh-poohs this alarm of Burnham's, and asserts that no harm has ever yet been shown to have come from them; but on the contrary, much good. He further suggests that they are unavoidable even in the absence of instruction, because they are the necessary outlet to the counting impulse, before names are grasped. At the same time, is this sufficient explanation for the fact that the number forms are extended to embrace not only the simple numbers but numbers over 100?

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[1] Howell: "Pedagogy of Arithmetic," basing his theory on the experimental work of Phillips.

[2] Hornbrook: "Pedagogical Value of Number."

[3] Burnham: "Cyclopedia of Education," Vol. I.

It would seem to be a difficult question to settle. Even by use of parallel classes, several considerations arise. Are the members of the two classes alike in number capacity to begin with; will the instruction itself create number forms where none exists; can children be found not already possessing such forms?

ORAL ARITHMETIC.

The claims of written arithmetic as against those of oral or mental arithmetic have long afforded grounds for quarrel. Before going into a brief discussion of the relative claims of the two, it may be well to point out the danger in the use of the expression "mental" as opposed to written arithmetic. The distinction has become so contrasted in the minds of some that they believe that written arithmetic involves no mental effort at all, and I think this often accounts for the tendency to urge the one as against the other. I was amused at an incident which occurred a few years ago during an investigation into the alleged unfairness of a certain paper in arithmetic set for one of the high school examinations. The investigation was presided over by a judge. One of the mathematical teachers interested was questioning the framer of the paper, and his complaint was that a certain question was merely a mental one. The answer came back quickly from the framer that he had intended all the questions to be mental. This caused a laugh around the court, in which the judge joined heartily. This well illustrates the perverted idea some have of the two phases of the same subject.

It is sometimes thought that oral arithmetic was an innovation due entirely to Pestalozzi. This, of course, is not true. It existed prior to the adoption of the Hindu numerals, for the very good reason that the system of numbers then employed did not lend themselves easily to written work, devoid as it was of zero and local place value as we have today. The new numerals lent themselves easily to "figuring," and in time oral arithmetic lost out, to be revived, as we have seen, by the Swiss reformer.

Oral arithmetic then became very popular on the continent and in America. This was due in part to the reasons which caused Pestalozzi to adopt it, particularly the rigid mental discipline it was supposed to furnish and also to the fact that much headway could be made in the subject without necessitating the purchase of scribbling paper, which was as far beyond the reach of the ordinary citizen in those days as it is today. With the advent of slates, oral arithmetic lost favour somewhat. The time, too, was ripe for a reaction against the over-emphasis of oral arithmetic. Since then, to within the last decade or so, the two forms of arithmetic have held out against each other with varying success.

It seems to me that the traditional quarrel between the two is a mere tempest in a teapot. No matter which phase is adopted, the mind must be strenuously active with worthy ends. The fact that in adult life practically all the everyday arithmetic is done without a pencil suggests that some facility in the use of reasonable numbers in the solving of problems, not too complicated, should be acquired, without the use of paper and pencil. Teachers should remember, too, that facility in oral arithmetic may not mean the same facility in written arithmetic; and *vice versa*. There would appear to be different abilities required.

In the introductory treatment of a topic where the numbers are small and the steps simple, oral arithmetic is most advantageous. It affords a quicker medium of discussion between teacher and pupils and gives the teacher easy opportunity to keep "tab" on the pupils' reasoning. As the numbers become larger and the steps more complex, greater reliance must be placed on the pencil. So that after a pupil has reached the stage when he reads, spells and writes, it becomes a matter of good judgment upon the part of both teacher and pupils as to the correct balance between the two. It seems absurd to place limits within which written or oral arithmetic, as the case may be, is to have a place. Throughout the entire course oral and written arithmetic should be complementary phases of the same subject.

Teachers in early grades, on account of the facility

the subject affords for seat work, load the pupils down with arithmetic in its written form. The danger is, especially in Grade I, that the teacher is neglecting those constructive exercises wherein number arises in the natural way, through which number concepts are developed.

A few minutes a day devoted to rapid oral work on the four processes help to fix the number facts which it is desirable the pupils remember. We are thus not overlooking the advantage in habit-building of learning a thing in as many ways as possible.

In all problem work, whether it finally involves pencil or not, pupils should be encouraged to weigh the data carefully in the mind before attempting a written solution. This will avoid the practice so common of trying to make the solution fit in with some type form the teacher has given some time previously.

DRILL.

No school practice has come into such bad odour in some quarters as the traditional drill. Undoubtedly, in the days of memoriter methods, drill was the teacher's chief instrument. All rules, principles, definitions, etc., were copied from the teacher's text or from the blackboard and committed to memory by the process of continual repetition. There was little if any rationalization in the methods of instruction.

As a new pedagogy emerged with the study of psychology, teachers saw that the old drill methods were wasteful and harmful, and, as is always the case, many went to the opposite extreme and banished all semblance of drill from the class-room. That this had its baneful effects is evident in the wide variations of performances in schools to which modern standardized tests have been applied.

Drill, in the sense of functioning experience as habit or the making of certain processes habitual or automatic, has a big place in school instruction. And it is the modern psychologists we must thank for giving to drill its rightful place in school-room procedure. If drill is a systematic endeavour to fix firmly habits or associations between stimuli

and responses, then the study of "learning" and "habit" and the laws based upon their study are necessary elements in every teacher's equipment.

Of the many factors involved in habit formation, repetition has been the one too often seized upon, indeed often the only one, with the result that drill and repetition become synonymous. This error, together with the discovery that drill was most effective in its initial stages, combined to make school life a drudgery for the young child. Little wonder that he positively hated school.

The psychology of learning has been well studied and set out by Professor Thorndike,[1] and will not be repeated here. The amount, rate and limit of improvement; factors influencing improvement; transfer of improvement; fatigue, etc., are all important to the teacher who wishes to improve the skill of her pupils in arithmetical computation. Certain factors, however, may well be mentioned here. These are the necessity of focalization of the process, repetition in attention and the resultant satisfaction. The more strenuously a process is focalized in its initial stages, the greater are the chances of its becoming automatic.[2] The more intelligence has been active in the acquiring of certain bonds, the less effort will be necessary in making these bonds lasting. Processes should be taught; the number facts should represent the pupils' own judgments. In teaching, there is a tendency to return somewhat to the days when the authority of the teacher was sufficient reason for the pupils. This is the "habituation *vs.* rationalization"[3] movement, and to me there are grave dangers ahead in its general acceptance. Pupils may be again reduced to the plane of trained animals. The principle is being generally accepted that drill should never precede an intelligent comprehension of the habit to be taught.

The need for concentration of attention during repetition introduces such questions as exclusion of distracting stimuli, time of day, length of drill periods, etc. All

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[1] Thorndike: "Educational Psychology; Briefer Course."
[2] Bagley: "Educative Process."
[3] Suzzallo: "The Teaching of Primary Arithmetic."

teachers know how difficult it is to concentrate when pupils are being distracted. The matter has been investigated experimentally by Vogt,[1] but there is little definite measurement available. Short drill periods are better than long ones, as teachers have found from experience. Mr. Brown, of Eastern Illinois State Normal School, substantiated this experimentally, using Stone's Standard Tests.[2] His own summary is as follows: "Five-minute drill periods upon the fundamental facts, preceding the daily lesson in arithmetic, were found to be beneficial in the sixth, seventh and eighth grades.[3] . . . The benefit was not limited to improved mastery of the number habits, but included efficiency in arithmetical reasoning. The improvement was still in evidence after the lapse of the twelve weeks' summer vacation. The benefit was most marked in the sixth grade and least marked in the eighth grade, although it was significant in all three grades." The advantage of short over long periods was also investigated by Kirby[4] in 1913.

The proportion of time allotted to arithmetic which is usually devoted to purely drill exercises was computed by Jessup and Coffman[5] from data received from city and county superintendents. The data showed that the median percentages of time given to drill work throughout the grades was as follows:

Grade	I	43	Grade	V	39
"	II	50	"	VI	31
"	III	52	"	VII	22
"	IV	45	"	VIII	17

It will be noticed that "drill" rapidly decreases in the higher grades. This seems to be in harmony with psychological findings which place the first six grades as the "drill" period of childhood, and is one of the reasons advanced for transferring the child at the end of Grade VI from the "elementary" teacher to the high school.

[1] Whipple: "Manual of Method and Physical Tests."
[2] "Journal of Educational Psychology," Vol. II, 1911.
[3] These were the grades tested.
[4] Thorndike: "Educational Psychology."
[5] Fourteenth Year Book of the National Society for the Study of Education.

It seems trite to remark that interest should pervade all drill exercises, and yet some teachers do not appreciate the necessity. The problem attitude should always be uppermost in the child's mind. If the drill is to be satisfying, the pupil must see that the desired process is worth while. This is entirely opposed to the practice not uncommon of giving arithmetic questions to the "kept-in" pupil to do.

Psychological as well as practical considerations urge the desirability of always teaching the most promising bonds. This is illustrated in the advantage of certain methods of subtraction over others. This is also opposed to the too common method of teaching the multiplication tables. The plan of teaching the simple combinations in serial order may be unpsychological as well as wasteful. There are certain combinations used very frequently by the pupil, and some which give more difficulty than others. Sufficient experimental work has not been done to decide just what is the best sequence. By following Dewey's plan of beginning with some large whole, more or less familiar to the pupils, as the number 12, suitable combinations could very easily be selected. Observation goes to show that there is favouritism among numbers. Four is said to be a general favourite. As a rule, odd numbers are unpopular with children.

In 1903, Sanford[1] used the data supplied by a guessing contest staged by a merchant in Worcester. Beans were placed in a five-pint bottle and placed in the store window. The person guessing the nearest to the correct number received the prize. Sanford went on the assumption that people would unconsciously name numbers that appealed to them. The results, which, with a doubt, indicated favourite guesses, are not of much interest in so far as they were adult guesses, but they afford further substantiation to the belief that children also have favourites.

Thorndike submits tentatively two factors in improvement, absence of emotional excitement and absence of worry. Undue competition in the acquiring of skills in computation should not be encouraged. The normal excitement caused

[1] Quoted by Howell: "Pedagogy of Arithmetic."

by success is ample. Competition should be directed towards beating one's own record. The practice of undue scolding or holding pupils up to ridicule for their shortcomings is without doubt negative in its effect upon the mastering of desirable facts.

Bryan and Harter showed that even after reaching a point of apparent limit, further practice may result in further improvement. Subjects may have reached a "plateau," but increased efforts may result in increased efficiency. Thorndike thinks that too often people are satisfied to remain at the efficiency of the plateau. Our limits, he says, are too often spurious, and physiological limits have not been reached at all.

It remains to be said that after the understanding of the process to be habituated, two courses may be followed, the so-called incidental drill or the systematic drill. By the former the teacher rather trusts to the later work of the school providing inevitable and sufficient opportunities for the use of the desirable process. There is a danger here that the "repetitions will not be sufficient to ensure automatism." By systematic drill the teacher may purposely provide opportunities in the regular work to ensure the necessary repetitions. This requires careful planning and at times may become artificial. Or the teacher may employ mechanical repetition, making the work as pleasant as possible by the employment of interesting devices and by occasional words of commendation.

It is evident that the first approaches the ideal. Someone has compared drill to viewing a city from various vantage grounds. "If the pupil could get all his drill while working problems which would interest him in themselves, the effect would be as happy as when he gets his physical exercise by romping in the school yard rather than by mechanically working with pulley weights." This is the ideal set up by Professor Young[1], and is well worth keeping before us.

We often hear the "old people" bemoan present day methods and tell how things were done when they went to school and how good were the results of the old memoriter

[1] Young: "Teaching of Mathematics," p. 216.

methods. In this connection the Springfield Tests are of interest. In 1905 or '06 certain old examination papers of 1846 in arithmetic, spelling and geography were found in the attic of an old school in Springfield, together with the answer papers. These questions were given to the corresponding grades in the Springfield schools and all sets of answer papers marked by the same persons. The average percentage obtained by the pupils in arithmetic was 65.6 as against 29.4 by the pupils of 1846. This may be partial evidence that methods of today, devoid of much of the old mechanical drill of the last generation or so, are more effective. All will admit however that much yet remains to be done.

HYGIENE OF ARITHMETIC.

The health of the child in relation to instruction in the various school subjects has been made a matter of recent concern with the result that many long established school practices are being brought into questionable light. Professor Burnham is the pioneer in this field; but the field has only been touched as yet and much remains to be done.

Undoubtedly the exaggerated notion of the value of arithmetic in the past allowed many practices to creep in, having efficiency as their aim, which are now known to be very pernicious in their effects on mental life. For instance, drill, *i.e.*, unpsychological drill, has many sins to answer for. Not only has there been a terrible waste from overlearning but there are positive ill effects upon intellectual power as Dewey[1] has pointed out. This overdoing of the mechanical and automatic restricts and warps reflective tendencies.

One of the first considerations is the question of fatigue. Assuming the existence of a curve of efficiency and of freshness, to which experience would point, such matters as: when to have the subject taught; length of lesson, proportion of time given to recitation and to seat work; distribution of play periods, etc., are all of great importance. The final word upon the question of fatigue has not been uttered, but Professor Whipple has well summed up the net outcome of

[1] "How we Think."

experimental work to date. This summary is given elsewhere in this thesis for those who wish to read it.[1]

Besides certain mental habits which are formed by education there are also certain secondary effects of instruction which it behooves the hygiene of instruction to take notice of. The importance of hygienic methods of instruction cannot be too strongly emphasized. The modern treatment of certain nervous and mental disorders by re-education is evidence enough of their importance.

In teaching arithmetic, the fact that individual differences exist, seems to be overlooked by the great mass of teachers. Not only may a child have low gifts for arithmetic, but he may be lacking in all round arithmetical ability. All sorts of variations of functioning in arithmetic may be found in one child. This neglect, as a rule, works to the disadvantage of the child of limited arithmetical capacity. And Dr. Sturgis[2] here finds, especially if the child is of nervous temperament, many chances for development of neuroses and chorea.

Dr. Triplett[3] enumerates many arrests, disorders, etc., resulting from faulty instruction in arithmetic. He mentions arithmomania, the counting habit, where the mind becomes filled with numerical relations to the exclusion of all others. He also refers to the persistence in consciousness of the coloured balls of the abacus with some children; dreaming about number tables, etc.; writing dollar signs by the hundreds, etc., as evil effects of bad methods of instruction. Number forms are also condemned by this writer in common with Burnham and others.

The age at which to begin the teaching of arithmetic is of course involved. This has already been dealt with from a different angle. From the standpoint of Hygiene of Instruction, Burnham quotes Patrick, "Mathematics in every form is a subject conspicuously ill fitted to the child mind. It deals not with real things but abstractions. When referred to concrete objects, it concerns not the objects themselves but

[1] Cyclopedia of Education, Vol. II, p. 581.
[2] Quoted by Burnham: "Cyclopedia of Education."
[3] Quoted by Howell.

their relations to each other. It involves comparison, analysis and abstraction." Professor Burnham, as we have noted previously, thinks that the formal instruction should be delayed much more than it is at present, although he submits no evidence to support his claim. He seems to be on the right track when he says that arithmetical work in the early years should be spontaneous activity on the part of the child, although he does not explain why this should not be characteristic of the work through all the grades. He is of the opinion that by thus postponing arithmetic it is possible to do away for the most part with many artificial and unpsychological methods which will exhibit their ill effects in later childhood. He adds that "the work in arithmetic should be simple, and the complex examples in logic and the like should be eliminated. In the case of nervous children special care should be taken to avoid worry and the development of neuroses and chorea. And, in general, special attention should be given to the secondary effects which are important from the view of mental hygiene."[1]

Because the pressure of arithmetic has resulted in many evil effects, we must not go to the other extreme and allow a life of ease in this subject. Well timed and proportioned vigorous mental work is always welcomed by the healthy mind and is a condition of its growth. Habits of listless mental work are as weakening as habits of lazy physical work, and this applies with equal force to all subjects. Upon the teacher rests the important duty, not only of carefully planning her methods of instruction, but also of teaching the pupils the best methods of study.

THE FOUR FUNDAMENTAL PROCESSES.

It was pointed out in Chapter I that the history of mathematics cannot be traced with certainty further back than the Ionian Greeks. The outstanding exception is the manuscript of Ahmes which carries us back to 2,000 years or more before the birth of Christ.

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[1] "Cyclopedia of Education," p. 209.

Part of this work we have seen is devoted to the fundamental operations and it is interesting to note that multiplication and division were treated as repeated additions and subtractions, respectively. All the operations were undoubtedly performed by some mechanical means, and this remained true up through the Greek period and through mediæval times up to the Renaissance when, with the introduction of the new numerals, operations were performed with the symbols themselves. Even today mechanical means are employed with the Chinese and other races, as we have repeatedly pointed out, while the invention of computing machines run by hand or electricity is quite modern.

The earliest plan of representing numbers in writing was by means of vertical strokes. Three vertical and of course parallel strokes represented our present 3. This same plan remained throughout Roman history, although separate signs were invented for 5, 50, 100, etc. The V of the Romans is supposed to be the hand with the thumb extended. X is of course two V's one inverted below the other. C for 100 is the first letter of the word centum meaning hundred. The L for 50 and the D for 500 are obscure in origin.

The Greeks used a somewhat similar plan at first but later (about 300 B.C.) modified it. The first nine letters of the alphabet were used to represent the first nine numbers. The tens from 10 to 90 were represented by the next nine letters, etc. This required an addition of several letters to their alphabet. By suitable arrangements of these letters, supplemented by a system of suffixes and indices, the Greeks were able to write numbers as high as 100,000,000. It is not surprising that the Greeks, with such a clumsy system, made little progress in the field of arithmetic. The operations were very clumsy and laborious. Multiplication and division were repeated addition and subtraction as with Ahmes. A table of multiplications learned by heart gave some relief to the work of multiplication. It is recorded how one Greek mathematician about 100 A.D., wishing to multiply 400 by 5, repeated 400 five times in addition, by means of the abacus of course. In dividing 6152 by 15 he tried all the multiples of 15 until he reached 6000. This gave him 400 and remainder of 152. He then began again with all the mul-

tiples of 15 until he reached 150, this gave him 10 and remainder 2.[1] The answer was then 410 with remainder 2.

Not all mathematicians used the method of repeated additions for multiplication. Hero of Alexandria multiplied 18 by 13 in the manner described below. It must be remembered that letters were used for the individual numbers however.

$$\begin{aligned} 13\times 18 &= (10+3)\ (10+8) = 10\ (10+8)+3\ (10+8) \\ &= 100+80+30+24 \\ &= 234 \end{aligned}$$

This last operation would be performed by means of the abacus.

With the introduction of algorism, or the Arabian arithmetic, based on the art of Alkarismi, and the consequent ousting of the old Boethian arithmetic, the abacus was pretty well discarded and operations were performed with the symbols themselves, following rules carefully set out in the commencement of all arithmetics. Even yet the operations were very cumbersome but improvements were gradually introduced, due largely to the Italian merchants to whom the new arithmetic appealed very strongly. The improvements consisted largely of the introduction of signs, logarithms and decimal fractions.

The origin of the various signs is not clear. Ahmes represented addition by a pair of legs walking forward, subtraction by a pair walking backwards, or a flight of arrows. In Greek times addition was indicated by simple juxtaposition, a relic with us being seen in such numbers as $2\frac{1}{2}$ which stands for $2+\frac{1}{2}$. Subtraction was indicated by a special letter. In India juxtaposition indicated addition, while a dot over the number to be subtracted meant subtraction. Then appeared at different times such forms p or $\overline{p}$ for addition and m or $\overline{m}$ for subtraction. The earliest instance of the use of + and − recorded is in the fifteenth century and the adoption was general by 1630. The origin of these signs is very doubtful. The sign = seems due to an Englishman who lived about 1557. Prior to his time the word was written in full.

[1] Ball: "A Short History of Mathematics."

For multiplication, a dot was used once, also juxtaposition. The present sign appeared in 1631. Division was early indicated by the existing form for a fraction. Our present sign would seem to be a combination of — minus, and : used to show ratio. The first use recorded was in 1659.

Coming now to the operations themselves, the only thing to note of interest in the history of the development of addition and subtraction is that the Arabs usually worked from left to right. The present plan of right to left became general about 1600 and the credit for its adoption belongs to one Gairth, an Englishman.

Subtraction has undergone some changes in the arrangement of work in performing the operation. Take for instance,

873
296.
577

There are four ways the result may be obtained.

1. 6 from 13 leaves 7; 9 from 16 leaves 7; 2 from 7 leaves 5.
2. 6 from 13 leaves 7; 10 from 17 leaves 7; 3 from 8 leaves 5.
3. 6 and 7 are 13; 10 and 7 are 17; 3 and 5 are 8.
4. 6 from 10 leaves 4; 4 and 3 are 7; 9 from 10 leaves 1; 1 and 6 are 7; 2 from 7 leaves 5.

The third is called the Austrian method and is preferred by some on account of additions alone being used. It is however condemned by Browne[1] and others. The last method is very old and not used to much extent.

In the field of multiplication and division the Hindus and Arabs found great difficulty. So laborious were the processes and so great the liability to error, that checks were invented through dire necessity. An instance of this is the rule for casting out the nines for multiplication. By inventing tables up to 5×5 and making other and higher number facts dependent upon these, the Italians were able to reduce the laborious features which faced the Easterners.

The difficulty of multiplication had led the Arabs to invent mechanical ways of effecting the process, and these

[1] "American Journal of Psychology," January, 1906.

were improved by Napier in 1617, by the use of Napier's rods.[1] These could also be used for division.

Division presented even greater difficulty than multiplication. The Arabs and Persians invented an unique method.[2] Later the Italians invented the famous galley or scratch method, probably the germs of which they got from the Hindus. The process of dividing 1,330 by 84, using the scratch method is described below. Where a number is underlined, it means that it is scratched out during the process. The completed process would appear thus:[3]

07
49
—
0590
—
1330(15
—
844
—
8
–

The process is as follows: first write the 84 beneath the 1330, as indicated below, then 84 will go into 133 once, hence the first figure in the quotient is 1. Now $1 \times 8 = 8$, which subtracted from 13 leaves 5. Write this above the 13, and cancel the 13 and the 8, and we have as the result of the first step

5
1330(1
—
84
—

which shows a remainder 490.

We have now to divide 490 by 84. Hence the next figure in the quotient will be 5, and rewriting the divisor we have

4
59
–
1330(15
—
844
—
8
–

[1] Ball: "History of Mathematics," where it is described.
[2] Idem. [3] Idem.

Then $5\times8=40$, and this subtracted from 49 leaves 9. Insert the 9, and cancel the 49 and the 8, and we have the following result

$$\begin{array}{r} 49 \\ - \\ 59 \\ - \\ 1330(15 \\ --- \\ 844 \\ - \\ 8 \\ - \end{array}$$

Next $5\times4=20$, and this subtracted from 90 leaves 70. Insert the 70, and cancel the 90 and the 4, and the result, showing a remainder 70, is

$$\begin{array}{r} 7 \\ 49 \\ - \\ 590 \\ - \\ 1330(15 \\ ---- \\ 844 \\ --- \\ 8 \\ - \end{array}$$

The extra zeros in the complete example first set out, are not necessary, but they do not affect the result, as it is evident that a figure in the dividend may be shifted one or more places up in the same vertical column if it be convenient to do so.

The scratch method was employed even after our present method was adopted, but it was finally discarded about 1700 in all countries.

There is an Austrian method of division which is illustrated in dividing 6.275 by 2.5:

$$\begin{array}{r} 2.51 \\ 25)62.75 \\ 50 \\ ---- \\ 12.75 \\ 12.5 \\ ---- \\ 0.25 \\ 0.25 \\ ---- \\ \\ 83 \end{array}$$

Here the decimal point gives no trouble and the entire remainder is brought down each time.

An introspective study made by Browne,[1] using adult subjects, has furnished much valuable information about the four fundamental processes. His results will, of course, require further confirmation before they can be accepted without question.

The four processes have their origin in counting. Soon counting is replaced by memorized tables, the so-called number facts of addition, subtraction, multiplication and division. The learning of these tables is usually through motor or motor-auditory response to visual stimuli. This motor response is linguistic, and in the early stages of addition the pupil must use a complete statement of the process. Thus to add, say, 4, 7, 3, etc., he must say 4 and 7 are 11, 11 and 3 are 14, etc.

Later he names the sums only, thus: 11, 14, etc. It should be the teacher's aim to rid the pupil of these tabular formulas as soon as possible so that greater speed may be attained. A child or an adult when confused, tired or uncertain, will lapse into this full verbalism.

When adding a column of single digits the partial sums are always given verbal expression, not often of course audible, and emphasis is always placed on the units digit of the sum rather than on the tens. This verbal statement of the sum apparently helps to fix the sum, subconsciously, and leaves the attention free for recognition of the next number to be added. After the first number is passed there are four distinct stages in adding: (1) the recognition of the sum, (2) the motorization of this sum, (3) the recognition of the next number to be added, (4) the associative process leading to the sum of the two. Stages (1) and (2) are the focal points of attention in addition and any weakness in either or in (4) retards the process and errors may result.

Of course success not only in addition but in all the processes depends upon the strength of the fundamental associations (bonds).

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[1] "American Journal of Psychology," 1916.

There are always certain special tendencies to error, due to incidental relations, order of sequence, etc. "When any preceding digit remains in or near the focus of consciousness (as often happens when there is uncertainty regarding the accuracy of a result, or, in case the attention has been called in a particular manner to a particular digit), such a digit is likely to displace or change the digit of the result which is, or should be, in the focus of consciousness at that instant."[1] For instance in adding 47 and 4 one subject wanted to say 11, because 4 and 7 are 11 got into his mind; another wanted to say 5 and 5 and 4 are 19.

A feeling that one has made a mistake will interfere with the subject's addition and cause errors. Occasionally associations fail absolutely and the mind becomes as it were a total blank. It is not always due to the mind wandering. The adder starts again with difficulty and henceforth is more or less confused and conscious of having lost time, all of which deteriorate results.

Throughout addition there is a feeling of certainty, which has much to do with effective work. If the adding goes too fast, slurring over results, this feeling of certainty is seriously affected. When the work is going too slowly, false associations suggest themselves, due probably to the attention wandering between its focal points.

The subjects found that even numbers are easier to add than odd, except such paired combinations as 3 and 5, 5 and 5, etc. In even numbers this ease is possibly due to the presence of the common factorial 2. Such combinations as 9 and 3 are easy for the presence of the common 3. It is hard to add even and odd digits, while a series of even digits beginning with an odd and thus having sums always odd, is still more difficult.

Interesting results followed the investigation of the relation of the size of the digit added to the difficulty of combining. It was found that the results of the study of Ebbinghaus of the memory for nonsense syllables applied to single digit numbers. This investigator had concluded, "that in the process of impressing any series of ideas upon

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[1] Browne: "American Journal of Psychology," 1916.

the mind by repetitions, bonds of association are formed between all the individual members of the series. Every member of such a series acquires a tendency to bring the other members along with it when it re-enters consciousness. These bonds or tendencies are of different degrees of strength. For remote members of the series, they are weaker than for neighboring members. The associative bonds for given distances backward are weaker than for the same distance forward. The strength of all bonds increases with the number of repetitions. But the stronger bonds between neighboring members are much more quickly strengthened than are the weaker bonds between more distant members. Therefore the more the number of repetitions increases, so much stronger become these bonds absolutely and relatively to those of more separated members."[1]

That is the law applied to the counting series means that in 9 + 2 = 11, say, the associative bond between 9 and 11 is comparatively strong, but between 2 and 11, as 2 + 9 = 11, it is very weak. "Applying this scale, two distinct tendencies appear for which the introspections afford a considerable body of evidence: (1) The easiest combinations will be those in which the greatest disproportion between the digits exist, as 7 + 2, 9 + 2, 8 = 3. In general, combinations will be harder in the increasing order of the smaller of two digits giving the same result; easier in the increasing order of the larger digit. (2) Continuing up the range of possible combinations with constantly decreasing differences to the point where the difference between the two digits combined is least, as 4 + 3, 5 + 4, 8 + 7, 9 + 8, the difficulty of combining should be greatest; and curiously enough, we seem to find here the same law of the shortest step also operative-adding by subtracting 1 or 2 in the case of 9's and 8's from the smaller of the digits, and saying the result in the 'teens, especially common in 9-combinations. With other odd and even digit combinations in this class where only a difference of 1 exists, as 8 + 7, 6 + 5, and much less consciously with 5 + 4, the addition is greatly reinforced and frequently comes directly from the doubling of the larger and subtracting 1. Thus

[1] Taken from Browne: "American Journal of Psychology," p. 9.

8+7=16—1, etc. It is easy to add 6+5 because the sum is just 1 short of the familiar doublet 6+6=12, and also 1 greater than the more direct 5+5=10. While 6+5=11 is comparatively easy, 7+4=11, as reported by the subjects, is most difficult."[1]

Below 10 the longest step is limited to four members (5+5 belongs to the 5-count and is easy) above 10 the maximum range is 8 members. As is the case below 10 so above 10, many of the combinations are easy. After the elimination of those which are relatively easy to any one, there is left a range of results 11 to 17 inclusive, derived by combining single digits, psychologically differentiated from the lower range in respect to the character of the associative bonds. Practically all adding depends upon the subconscious recapitulation of the range 1–10 and of the 'teens. The tens remain as a subconscious count, it being with the digit relations in this space that the adding psychosis has to do. Any difficulty one has in the 'teens will crop up later in all column adding. Thus if one has trouble with 7+6 he will have difficulty with 27+6, etc.

Multiplication is historically repeated additions of equal numbers. It is a short cut but a more highly developed process and more difficult than addition.[2] Somewhat the same factors as were present in addition are here also. Any lack of promptness of recall of the desired product leads to a more decided motorizing of the delayed result, there occasionally being a full restatement of the verbalism involved. Undue delay of association results in reversion to counting.

As one would expect the multiplicand is in the focus of attention, the multiplier being an almost unconscious element in the process, but of course is important in putting the mind in the proper "set" of associations.

In written multiplication the carrying is the most vulnerable part, and the source of many errors, due probably to the fact that the conditions of normal adding are reversed, the digit to be added coming first, the product to which it is to be added second.

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[1] "American Journal of Psychology," 1916, p. 10.
[2] McLellan and Dewey: "Psychology of Number."

Browne suggests that the multiplier should be the smaller number, for a matter of economy. Thus to multiply 3 by 9 would simply mean reversing to 9 by 3, and the tables would be thus cut down considerably. He also suggests separate tables be made for the paired cases, such as 9×9, 8×8, etc.

In both addition and multiplication the processes are synthetic, the bonds operating forward. "In subtraction and division the bonds of association operate backward, hence the two distinctly different classes into which the four simple processes divide."

In subtraction two exercises were tried, one the school room practice of continued subtraction as, 100—7—8—5—9, etc., the other written subtraction. All the subjects reported that it took longer to initiate the subtraction "set" or attitude than in the case of addition or multiplication.

In the continued subtraction the same four stages were noted as are found in addition. The same laws are operative except in the reverse order, which makes subtraction a decidedly harder process than addition, associations backward being weaker. This explains why in subtraction, especially when the subtrahend exceeds the minuend, the additive process is employed rather than direct association. Persons translate the backward associations of subtraction into the forward ones of addition. So that while it may be easier, that is, more primitive, it is a less economical procedure than using direct associations. Persons employing the additive procedure, when they strike a weak additive bond, may slip over into the direct subtraction. For example, with many people $3+9=12$ is a weak association, so that in subtracting 12—3, they employ a direct subtraction bond. In case of 12—9, very likely many people translate into the additive bond.

The forward associations being first formed are the strongest, so that after one has acquired in subtraction the more economical direct subtractive associations, he finds himself loath to trust these and proves by the additive bonds. If subtraction facts were as well learned as the additive, there would be no need for this double procedure.

It would seem that the Austrian method of subtraction is not the most economical where one could be well grounded

in the basic subtraction associations. Pupils should be given practice in counting backward so as to have the same facility backward as they have forward. This would eliminate many of the troublesome subtractive bonds. Browne is of the opinion that the older method of increasing the subtrahend is superior to the present method of increasing the minuend, in that the difficulties of borrowing would be eliminated.

In division many of the phenomena observed in the other processes appear here too. At first, multiplication formula appears, but gradually the direct associations are used. In fact, division does not revert to multiplication for verification nearly as frequently as subtraction does to addition.

In written division there are numerous partial dividends to be dealt with which are not always exact multiples of the divisor. This fact, together with the multiplication and subtraction operations to be done, make the division process rather complicated. "The difficulty of the process as a whole increases with the size of the divisor, because of the increased range of possibilities as to the dividend numbers falling above the multiple."

In all the processes all the subjects laid importance on the act or attitude of mind which directs the particular process and holds it to the proper field.[1]

When educators had to recede somewhat from the extreme view that arithmetic provided a general training of intelligence, which could be directed into any other channels, some fell back on the assertion that there is transfer among the various operations of arithmetic. This has led to considerable investigation and the conclusions are of prime importance to the teacher.

Probably Stone[2] was one of the first to enter the field and after a number of experiments concluded, among other things, that there was no such thing as arithmetical ability, but that arithmetic was a complex of abilities made up of several specific abilities relatively distinct, a low degree

[1] The importance of this set or attitude is discussed by Thorndike "Educational Psychology."

[2] "Arithmetical Abilities and Some Factors Determining Them"

of correlation existing among them. The results of this investigation have been reinforced by those of others, all pointing to the fact that there exists a wide range of individual differences in capacity and a greater or less independence of each other on the part of the different abilities involved.

There are, in fact, as many different abilities as there are types of examples. One may easily test himself by adding columns of, say, 3 figures, as against columns of, say, 12 figures, to see that different abilities are required. This particular phase has been followed up by Mr. Courtis, who has identified and set out the various types of examples in the operations with integers alone.[1] The field of fractions has not been studied to the same extent. Each type requires a specific habit. There will, of course, be common elements in many of the types, but there is always sufficient difference to enable a person to be efficient in one type and less efficient in another. The important characteristics of the various abilities are: rate of performance and accuracy of response.[2]

Winch, in 1910 and again in 1911,[3] undertook experiments to solve the problem "Does Improvement in Accuracy of Numerical Computation Transfer to Arithmetical Reasoning?" Parallel classes were used. One class was practised in computation, while the other was given other work entirely. After a certain time both classes were tested in problems involving reasoning. In marking the papers, notice was taken only of the reasoning, the accuracy of the computation being neglected. The conclusion reached was that improvement in accuracy of arithmetical computation does not show any improvement in accuracy of arithmetical reasoning.

In 1911, Starch[4] made a somewhat similar investigation. "Eight observers practised for 14 days on mental multiplication. Before and after the practice they were given six tests in arithmetical operations and two in auditory memory span. For comparison, seven other observers

[1] See Appendix E.
[2] This subject is continued further, p. 114 *et seq.*
[3] "Journal of Education Psychology," 1910 and 1911.
[4] "Journal of Educational Psychology," 1911.

were given the preliminary and final tests without the practice series. The practised observers showed from 20 to 40 per cent. more improvement in the arithmetical tests than the unpractised observers. There was little change in memory span for either group.

". . . . The improvement in the end tests was due therefore to the identical elements acquired in the training series and directly utilized in the other arithmetical operations. The two main factors were (*a*) an increased ability to apprehend and hold the numbers in mind and (*b*) the acquisition of the ability to visualize arithmetical operations."

There would seem to be a discrepancy between the results of the two investigations, but it is only apparent. In the first there is no relation whatever between the processes tested, and consequently no transfer. Such is not the case in Starch's. This would seem to bear out the view that transfer is proportionate to the common elements involved.

After outlining certain experiments of Wimms, to determine the relation between improvability, retentiveness of practive effects, fatiguability and output of work in the two apparently related processes of addition and multiplication, and which indicated that these were not the same for any one subject in the two processes, Dr. Myers concludes: "For subjects of school age there can be no doubt that the two tasks differ materially in character."[1]

Arithmetic, then, should no longer be regarded as one subject. A pupil may not be either good or bad in arithmetic. There is no special arithmetical faculty. The subject embraces a series of operations, some more or less related, others completely independent.

On account of the many activities of mind involved in arithmetical computation, computation-tests have been employed widely to measure not only associative processes but mental efficiency at large. An example of a test for controlled association may be seen in Whipple,[2] while the more general use of arithmetic tests for general intelligence may be gathered from any standard work on Intelligence Tests.[3]

[1] Myers: "Experimental Psychology," p. 117.
[2] Whipple: "Manual of Mental and Physical Tests," p 460.
[3] For instance Terman: "The Measurement of Intelligence."

CHAPTER V.

SOME ASPECTS OF NUMBER TEACHING.
(Continued.)

PROBLEMS.

Growing naturally out of the point of view we hold as to the place of arithmetic in our schools, our contention is that the problem is the centre of the whole work. The mechanical work is, after all, only incidental. The healthy school is continually arousing the problem attitude in the children, in connection with ends worth striving for; and in solving these problems and realizing these ends the child is genuinely thinking, unsupported by artificial props to bolster up his interest. Under the old methods far too much time was spent on the bare mechanical tools of the subject, rather than on the content. With the problem as the centre of interest, a wholesome connection is made between the school and the pupils' varied interests.

It is generally recognized as a sound pedagogical principle that all knowledge to be real and permanent must be founded upon and developed out of the individual's experience. In order to have arithmetical ideas, mental images must be called up and these must be products of experience. Without this clear mental imagery the pupil will have no success in solving problems.

This means then that the problems must be "child," not "adult" problems. The problems must arise out of their play, their games and constructive activities and the community life of which the children are a part. The problem thus becomes real and concrete. It may be unnecessary then to add that the child must understand the situation or conditions which give rise to the particular problem.

Naturally this places considerable limitations upon the use of the text-book. The teacher should not become a slave

to it, but treat it as a guide and reference. Even with the eliminations already suggested there will never be a perfect text. It must be different for each locality. The great reservoir which holds the necessary material is the school and community life. The thoughtful teacher will use the text-book very discriminatingly and supplement it liberally.

Experience has shown that pupils' failures to correctly solve a problem are due to one or more of the following:

1. Difficulty of interpreting the language employed;
2. Error in the mechanical work;
3. Failure to reason out the various steps.

There is a very close relation between number 1 and number 3. Every teacher has noticed the close relation between reasoning in school problems and the language involved. Miss Rogers[1] has shown how this is true also of more advanced mathematics. "The ability to understand sentences, to conceive clearly the meaning of a given problem, is as important an element in its solution as any connection it has with algebraic symbols or their manipulation."

Errors frequently arise from the fact that the basic number relations have not been sufficiently habituated. It is but one instance of the neglect of the school maxim, based on the law of habit: Put together what you wish to have go together. Reward good impulses. Too often when once a desirable connection has been initiated, for example, 4 and 3 are 7, the opportunities given the child to use this are not sufficient to make the connection permanent. Frequent drills should be helpful in this connection. The whole question is one largely of habit formation and teachers would do well to familiarize themselves with this phase of psychology. Where the errors are the result of careless work the habit of checking and proving the work should be instituted. In fact, this is most valuable in any event and is part of the larger habit of self-correction developed in the best schools. From Grade I, pupils may be taught to check their work and where possible apply independent proofs.

Failure to reason may be the result of indolence on the part of the pupil. Pupils often try by means of specific words

[1] "Tests of Mathematical Ability," etc.

in the problem, to recall processes employed on a previous occasion rather than put forth the effort to think out the solution. A slight change in the wording, without otherwise affecting the problem, will put many pupils entirely at sea. A more likely cause is the lack of familiarity with the conditions under which the question arose; in other words, the pupils have no concrete background from which to reason. This is especially true in the upper grades where pupils are left to gather for themselves, from the text, their first ideas of some new topic, such as commission. This condition is frequently found in ungraded schools where the number of classes precludes any attempt of the teacher to give the time she would to the individual classes. It would amply repay any teacher, however, if she were to take time to discuss carefully with the class every new topic in arithmetic. The problems which follow should be natural applications of the principles taught, they should be of gradually increasing complexity and should present no unreasonable difficulty. My experience has been that pupils waste many hours a week lolling along over the arithmetic text with no idea at all of the subject about which the questions centre, occasionally solving, in a way, a problem by means of the answer in the back of the text.

The following mode of attack on a problem is suggested, to be employed often enough with the pupils until it becomes a habit of attack with them:

(1) Pupils read over the problem aloud (later on, silently of course);
(2) Pupils tell what the problem says in their own words;
(3) They tell roughly how they propose to work it;
(4) They estimate the answer;
(5) They work out the solution on paper;
(6) They prove their results.

At any step the teacher is free to question and put the pupils back on the right procedure in case they have gone astray.

One of the most important phases of problem work is problem-making by the pupils themselves. The problem attitude should permeate all school work, and especially that

of arithmetic. Problem-making and problem-solving are foreign to many schools and yet nothing will vitalize school work more. Teachers who have not been in the habit of raising this problem-making attitude will be surprised at the inability of their pupils to formulate a problem arising out of a simple concrete situation. I read some time ago a description of an experiment which will illustrate how weak pupils are in this matter.[1] The investigator took 1,300 children of Grades III to VIII one at a time into a room and showed each a tally register with a number already registered. The pupil was asked to record the number on a slip of paper. He was then told that the operator would work the tally for a minute registering additional strokes. At the end of the minute the pupil was to look at the register, take the number and calculate the number of additional strokes. The percentage of children who put the larger number above the smaller, already on the paper, and subtracted, was: Third grade, 35; fourth grade, 68; fifth grade, 54; sixth grade, 47; seventh grade, 97; eighth grade, 93.

We see a beginning in the formulation of problems in the case of pupils in Grade I, when the teacher asks for a story about the number 7, say, but this is seldom carried on through the grades and recognized as important. This problem-making on the part of the pupils is but a particular case of large areas of motivation, purposeful activities, or the project method, and will be discussed from the standpoint of motivation, somewhat further, under that title. This again is in keeping with the new aim of socializing all school instruction.

A very noticeable weakness with school children is their inability at analysis. Inspectors of schools continually point this out to me. This weakness accounts for the failure to reason out a problem and solve it correctly, and extends into the high school as we know from actual experience. Each year in the work in arithmetic with our teachers-in-training it is my custom to write an ordinary business problem on the board, often one in bank discount, and ask the students to outline the situations to which the

[1] Scott: "Social Education."

problem would apply. The response is always very small, and a few, more frank than the rest, will admit they never understood such a problem before, and yet had been working similar ones for years. This suggests that teachers should do more along the line of problem interpretation.

There is some difference of opinion as to where problem work should be introduced into the school course. It is not unusual to find teachers giving the early grades, say, the first five, over to mechanical work almost exclusively, and emphasizing problems in the remaining grades. While I agree that the mechanical work should be covered in the first six grades fairly completely, the drill period as it is called, yet in my opinion the problem is the pivot upon which arithmetic, so far as it is a school subject, hinges and must consequently find a prominent place all through the grades. Before pupils reach school they have their simple problems involving dolls, marbles and toys generally. If school is not to be divorced entirely from out-of-school experience, then, such problems must form part of school work, and the character of the problems must keep pace with the ever-expanding experience of the children. Mention has already been made of the play instinct and dramatization, both valuable in the lower grades.

A protest should be raised against the insistence of some teachers upon the exclusive use of the Unitary Method. It certainly has gone to the extreme when a Grade VIII pupil is compelled to solve a simple interest question as: Find the interest on \$450 for 430 days at 8 per cent. per annum, in the following way:

Interest on \$100 for 365 days is \$8

" " \$100 " 1 day " $\frac{\$8}{365}$

" " \$100 " 430 days " $\frac{430\times\$8}{365}$

" " \$1 " 430 " " $\frac{430\times\$8}{365\times100}$

" " \$450 " 430 " " $\frac{450\times430\times\$8}{365\times100}$

etc.

This brings up the larger question of the teacher forcing his solution upon the child in any event. Children should be encouraged in their own solutions if they show good reasoning. The point may be illustrated by the following solutions of the problem:[1] Find the cost of six oranges, if four oranges cost twenty cents. Any one of the solutions should be accepted.

(1) If 4 oranges cost 20 cents

1 orange will cost 20 cents $\div$ 4, or 5 cents,
6 oranges will cost 5 cents $\times$ 6, or 30 cents.

(2) Since there are 4 oranges, the cost is divided equally into 4 parts, thus placing in each part 5 cents. When there are 6 oranges, the cost will contain 6 parts, which will require 5 cents $\times$ 6, or 30 cents.

(3) 6 oranges equal 4 oranges and $\frac{1}{2}$ of 4 oranges, hence cost equals 20 cents and $\frac{1}{2}$ of 20 cents, or 30 cents.

(4) 1 orange is $\frac{1}{4}$ of 4 oranges, therefore cost of 1 orange is $\frac{1}{4}$ of 20 cents, or 5 cents.
6 oranges are 6 times 1 orange, therefore the cost is 6 times 5 cents or 30 cents.

(5) 6 oranges are $\frac{6}{4}$ of 4 oranges, therefore the cost of 6 oranges is $\frac{6}{4}$ of 20 cents, or 30 cents.

"SPECIAL" METHODS.

There will be considered here, briefly, some of the special methods in arithmetic which have had more or less popularity. The methods of Pestalozzi, Grube and their followers have already been discussed.

We often hear the expression, the logical method, and we have had occasion to remark that certain steps while logical were not psychological. Logical refers rather to the orderly arrangement of the adult mind, psychological to the workings of the child mind. It should not be overlooked that ultimately the goal of the psychological is the logical.

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[1] From the Ontario Teachers' Manual.

To the adult, the logical is psychological, while to the child the psychological is the logical.

The logical method characterized the traditional teaching. As a result of child study, the opinion came to be held that the procedure in teaching should conform more to the child's outlook and experience. Instruction would thus mean more to the child eventually. In the final outcome, of course, the child must see his experience set in scientific and logical arrangement. The point is that the child and the adult do not look upon life and the things of life in the same way.

Pestalozzi's great ambition was to psychologize education and he went a long step in that direction. Too often, however, looking at it from this distance, he lapsed into the very methods he was trying to avoid. Grube no doubt thought he was strictly psychological, but we are convinced now that most of his treatment was rigidly logical. Illustrations are, his simultaneous treatment of the four fundamental processes and the thoroughness demanded at all stages. Modern methods are not without their logical practices; for example, the insistence by some teachers of type forms of solutions. The teacher thinks his method the one and only suitable one, and the child must fit his reasoning into it.

The spiral and concentric methods have had their advocates. These methods would treat the fields to be covered, several times, each successive treatment being an expansion of the previous one. The spiral method calls for a more continuous treatment than is implied in the concentric, otherwise they are the same. The Grube method of covering the space 1 to 10, in Grade I; 1 to 100 in Grade II, etc., is an example of these methods.

The topical method is one used more frequently in history, geography and composition than in arithmetic. The idea is to select a topic of wide interest to the pupil which will motivate the work for some considerable period. Great care is required in the use of this method, but the field of arithmetic lends itself easily to such treatment, combined with the spiral or concentric methods.

The ratio method would make the ratio idea explicit from the first. It is the idea behind the Speer[1] method.

[1] Speer: "Speer's Arithmetics."

Undoubtedly number is a ratio. Against giving this idea early prominence is the historical development of the race, also psychological considerations, which would seem to bear out the contention that the idea is too abstract for the young child.

Speer would bring this idea into immediate consciousness by acts of measuring using solids, etc. All through, measuring is used in a narrow sense. He does not recognize counting as a form of measuring. This is recognized, however, by such writers as McLellan and Dewey. By using objective material in a natural and constructive way, the child gets ample opportunity to count and measure, with the ratio idea implicit all the time, but not forced upon him. Finally, when the work of fractions is studied the ratio idea stands out in clear light and the pupil has grasped the true essence of number.

These are the most important special methods and teachers will probably recognize in their own methods traces of them all.

It has been suggested that the one-to-one correspondence of higher mathematics finds simple application in teaching the early numerical ideas. Thus to the group : : there is the name "four" and the symbol "4." Teachers who have adopted the suggestion find it very helpful.

TIME DEVOTED TO ARITHMETIC.

When teachers still persist in devoting from ninety to one hundred and five or more minutes a day, and the best part of the day at that, to arithmetic, in a time when the cry is going up of a crowded curriculum, it behoves teachers either to justify their practice or make a radical change. No real defence is ever offered for this apparent time waste. That the majority of teachers give this amount of time to the subject was again brought to my attention during a three weeks' special inspection of some 30 rural schools this spring.

When teachers are questioned for reasons, various replies are given. Some plead the importance of the subject, so great that they make it the sole basis for promotion very

often. Others say they cannot "get around" their classes in less time. But one notices that the subject does not cease after the teacher has reached and passed a particular class, but continues to the first intermission. Many confess they have no reason except this is what they did in their own school days. The short training at Normal School has not been sufficient to uproot habits of years, with the result that teachers go out and lapse into the old ways.

It is evident, too, that teachers find arithmetic a satisfactory subject to teach on account of its results lending themselves, apparently, to tangible measurements. It is also one of the easiest forms of seat work, or better "busy" work, for the lazy teacher.

The schools of the past have always given this subject the pre-eminent place. This is due partly to the impetus which Pestalozzi gave to its instruction and partly to an exaggerated idea of its practical importance and its value as mental discipline. To these reasons must be added the fact that until of late years, the number of subjects was small and nothing was crowding for admission. It is only within the last 35 years or so that anyone has ever questioned the pre-eminence of Arithmetic.

With the enriching of the curriculum, came the necessity of curtailing somewhere. An examination of the content of the curriculum showed clearly that much dead and useless matter was being carried over from earlier times, matter which at one time had quite a legitimate place. Further considerations led educators to the opinion that the utility value of arithmetic was being greatly over-rated, while from another quarter the doctrine of formal discipline was attacked. In the field of secondary education the dissatisfaction was crystallized in the report of the Committee of Ten, who, as we have seen, were unanimous in the opinion that the course should be abridged and enriched. The Committee of Fifteen a few years later reported in part, "Your Committee believes that, with the right methods and a wise use of time in preparing the arithmetic lesson in and out of school, five years are sufficient for the study of mere arithmetic—the five years beginning with the second year and ending with the close of the sixth year. . . ."

As far as the much advocated utilitarian value is

concerned we found that it was much over stated. As far as the ordinary person is concerned, the requirements are not extensive and are fairly well covered in the following: accuracy and rapidity in the four fundamental rules embracing integers, simple and vulgar and decimal fractions; familiarity with a few tables of denominate numbers; ability to work easy problems in interest and commercial discount; the simpler phases of mensuration.

Recognizing the limits of formal discipline, it seems useless to drill upon any subject beyond the point when it ceases to furnish ideas useful to us in life or future study. There should be fairly positive reasons for departure from this suggestion.

If further support for this contention is necessary, one need only turn to Dr. Dewey's address "Waste in Education." From the standpoint of the child the great waste in the school comes from his inability to utilize the experiences he gets outside in any complete and free way within the school, while on the other hand, he is unable to apply in daily life what he is learning in school. When he enters the school he finds an entirely new world and the work of the school too often attempts to prepare for a life that never is.

In this connection it is worth while to refer to a study made by Jessup and Coffman on the "Economy of Time in Arithmetic." Data were secured from a questionnaire sent to city and county superintendents in the United States. The questions asked covered such topics as, elimination, additional emphasis, time devoted to drill, distribution of time among grades, etc. The summary, so far as it affects elimination of topics, as given by the authors is: "This study reveals the fact that there is an overwhelming tendency on the part of half the superintendents in this country in favour of either eliminating or lessening the attention to be given to such subjects as alligation, cube root, unreal fractions, progression and certain obsolete tables such as folding paper, surveyors' tables, etc. Again it reveals an overwhelming attitude in favour of increased emphasis on such fundamental subjects as addition, multiplication, substraction and division. There is also a decidedly strong disposition to favour increased emphasis on the application of arithmetic to the social and economic conditions of the day, such as the

saving and loaning of money, taxation, public expenditures, life insurance, etc."

We have tried to emphasize the fact that much time is wasted by too loose relation between what is done in school and what is done outside. We must now consider what time should be devoted to the subject.

The Committee of Fifteen suggested 60 minutes a week for Grades I and II, the work to be oral; and five periods a week in the other grades to the end of VI, where arithmetic was to cease. Periods should run from 15 minutes in Grade I to 25 minutes in Grade VI.[1] Payne,[2] in drawing up an ideal daily programme, suggests 12.5% of the whole time for arithmetic. This would mean roughly 35 to 40 minutes a day. Payne made a comparison with the average from ten typical American cities which gave 17.3% to arithmetic or roughly 50 minutes a day. While Payne's figures are meant for city schools, they are none the less suggestive for rural schools as well.

The study of Jessup and Coffman, referred to a moment ago, also furnishes valuable data for the point now under discussion, and when occasion arises I shall quote the authors' own words. The reports were secured from 630 superintendents both city and country. The time devoted to arithmetic varied in all grades from 0 minutes in the first three grades to over 300 in all grades per week. There were 136 schools, for instance, where no time was allotted to Grade I arithmetic; 13 gave 25 minutes; 103 gave 75 minutes; 108 gave 100 minutes, etc. "Interesting figures occur in all grades but we can only deal with averages. The median time spent in the first grade for the United States as a whole is 75 minutes; for Grade I, 150 minutes; for Grade III, 125 minutes; for Grades IV, V, VI and VII, 150 minutes; for Grade VIII, 165 minutes. (24 schools had no time for arithmetic in Grade VIII.) This means that one-half of the cities spend this amount of time or more in the various grades, and one-half this amount or less. The chart (not shown here) shows that one-quarter of the schools spend 25 minutes or less in the first grade; 75

[1] "Report of Committee of Fifteen," page 66

[2] Payne: "Elementary School Curricula" as reported by Bagley: "School Management."

minutes or less in Grade II; 100 minutes or less in Grades III and IV; 135 minutes or less in Grades V and VI; 150 minutes or less in Grades VII and VIII. The same chart shows that another quarter spend 100 minutes or more in Grade I; 125 minutes or more in Grade II; 150 minutes or more in Grade III; 200 minutes or more in Grades IV, V, VI, VII and VIII. It will be seen then that some schools spend relatively far more time than others on arithmetic. If one-quarter of the cities can get satisfactory results with an expenditure of from 5 to 20 minutes per day or less in Grades I to IV, there is reason for inquiry as to the accomplishment of cities which spend from 20 to 40 minutes or more per day during the first to fourth grades. Again, if one-fourth of the cities are able to get satisfactory results in from 20 to 30 minutes per day or less in the fifth to eighth grades, certainly we have cause to question the reason why another fourth of the cities spend from 40 to 60 minutes or more per day in these grades. . . . In other words, from these investigations, we have no reason to expect that City A, which gives twice as much time as City B to arithmetic will get results in the same proportion."

The findings of Dr. Stone[1] in 1918 are more definite. Dr. Stone studied the arithmetical achievements of children in the sixth grade in 26 school systems. The tests embraced not only mechanical work but reasoning also. His statistics show that a large amount of time expended is no guarantee of a high standard of abilities. There is practically no relation between time expenditure and arithmetical abilities. He also concludes that ability to handle such foundational work as is measured by the tests in this study, has not necessarily suffered by the introduction of other subjects and the consequent reduction of its time allotment. He also finds that the influence of home is not responsible for difference in abilities, the differences in the main being explained by teaching and supervision.

No investigation is necessary to determine "when" arithmetic at present comes in the daily programme. The universal practice is to assign it the place between the

[1] Stone: "Arithmetical Abilities and Some Factors Determining Same."

opening of school and first recess. Of course it is often continued after recess and repeated in the afternoon session.

It has always been assumed that arithmetic should have the best place on the time-table, on account of its difficulty and fatiguing qualities. Educators have always been able to draw a "work curve"[1] showing that pupils' efficiency reaches its highest point between nine and ten o'clock in the morning, declining rapidly to a minimum at noon. In the afternoon, a high point is reached about two, but is lower than the maximal morning point. Having agreed upon this curve, the next step was to determine the relative order of the subjects, from the standpoint of fatigue production. Bagley and others[2] have found mathematics one of the most fatiguing subjects, which seems to bear out common experience of teachers. Arithmetic has consequently occupied the time on the daily programme when children are supposedly the freshest.

The question of fatigue has undergone considerable experimental investigation of late years and there is by no means unanimity.[3] Thorndike[4] is of the opinion that if it is made worth while a pupil can work as well at one time of the day as another, and this opinion is borne out by experiments made by himself, as well as data from other investigations. If Thorndike's contention is sound, modifications in the traditional time-table will be in order.

MOTIVATION.

We owe much to Rousseau for bringing into clear light the doctrine that the child, his interests and needs, is the centre of the educative process. Pestalozzi, Froebel, Dewey and a host of others, have emphasized the same doctrine. This growing realization of the importance of childhood has been at the bottom of most of the school reforms of the last hundred years. No longer is the child regarded as a miniature adult. His childhood is a necessary and important

[1] Bagley: "Class Room Management," also "Educative Process."
[2] German psychologists particularly, see Sandiford: "Mental and Physical Life of School Children."
[3] Sandiford, also Rusk: "Introduction to Experimental Education."
[4] Thorndike: "Educational Psychology, Briefer Course."

period of human life. The more fully it is realized the better adult will the child become.

Within this larger question is bound up the question of relating the content of the curriculum to the child's needs and experiences. Is the school work of real significance to the child? In too many cases it is not, resulting in indifference of many children toward school and in many cases positive hatred. We are beginning to realize more than ever that it is not always economic pressure that drives the boy out of school, but failure to enlist the interests and energies of pupils in the work of the school. And even while we may compel physical attendance it does not follow that the child is present mentally.

Professor Suzzallo[1] says: "By good teaching is meant provision of school experience wherein the child is wholeheartedly active in acquiring the ideas and skill needed to deal with the problems of his expanding life. . . . Events must happen to him, in a way to bring a full and interested response. . . ." Schools too often go on the assumption that the accumulated learning of adults is the proper material for the child. If he takes to it kindly, all well and good, if not, his future welfare demands that it be forced upon him. The child is not opposed to learning but he learns, as does the adult, through dealing with real situations. ". . . So also in the cultivation of the thinking power of the child it is more important at the beginning that he actually thinks, actually deals with situations, which are consciously problematic,[2] than he should think in the most finished form." In the case of children these situations are child situations. Rousseau, crying against the haste of the schools to turn children into grown-ups, exclaims, "Do not save time, but lose it. If the infant sprang at one bound from its mother's breast to the age of reason, the present education would be quite suitable . . . . Nature would have children be children, before they are men. If we try to invert this order we shall produce a forced fruit, immature and flavorless, fruit that rots before it can ripen. . . . . Childhood has its own ways of thinking, seeing and feeling."

[1] Suzzallo in Introduction to Dewey: "Interest and Effort."
[2] Miller: "Psychology of Thinking."

It is this appreciation of childhood as a definite period of the total of life with its peculiar materials for, and mode of, development that has resulted in emphasis upon motivation of school work. "That attack upon school work which seeks to make its task significant and purposeful to each child, by relating it to his childish experiences, questions, problems and desires is called motivation. The child's work is motivated whenever he sees a real use in it, whenever it satisfies some need he feels, provides some value he wants, supplies some control he wishes to possess, secures some desired end, or helps him to attain any definite goal." And this goal may be near or remote. ". . . . What the schools need, and what is proposed through motivation, is a change in the inner spirit of the school, a change that shall recognize the child as the centre of the school's efforts, giving subject matter a secondary place. . . . . In the elementary schools, administrator, teacher, course of study, and the inner spirit of the school must become subservient to the child, his interests and the problems that to him are vital."[1]

Motive is thus bound up with the notions of interest and effort, and to Dewey we must look to see these in their true light and relation. Professor Dewey[2] has given a true conception of interest as the accompaniment of the identification through action of the self with some object or idea for the maintenance of a self-initiated activity. The conflict between the doctrine of interest and that of effort is still alive. Dewey shows that, however opposite the arguments may appear they are nevertheless based on a common assumption, namely, that the object, idea or end to be measured is quite external to self. On this assumption the advocates of interest say we must surround the object with as many pleasant features as possible—sugarcoat it; while the opponents say only a "slow dead heave of the will" will secure the desired end. The true conception of interest identifies the fact to be learned with the growing self, and is absolutely necessary if the person is to grow and develop. If this identification is once secured, no appeal to extraneous

[1] Wilson and Wilson: "Motivation of School Work."
[2] Dewey: "Interest and Effort."

devices is necessary. The current idea of effort would suggest a separation between the self and the fact to be mastered, resulting in divided activities which would be fatal to growth and realization of self. "Howsoever winning the work or recitation, at times in his private preparation the tide of the pupil's interest will be on the ebb. In that hour of trial only a vigorous will, capable of pursuing the uninteresting though important task, will prevent the pupil from being beached on the shore of inactivity and idleness. Effort is the strain consciousness puts on itself in performing unattractive work. It is voluntary attention to the uninteresting. It is the will to do one's duty when one doesn't want to. . . ."[1] But rather does effort arise in the attempt to allow full development to certain powers within the child which are ever ready for expression. Dewey says: "Adequately to act upon these impulses involves seriousness, absorption, definiteness of purpose; it results in formation of steadiness and persistent habit in the service of worthy ends. "And this effort is permeated by interest because the self is concerned throughout. It can never degenerate into drudgery or mere strain of "dead lift." The consequence is not a division of activities but a unified activity.

The old phrase, "Making things interesting," really means then that we must select and present material relative to the child's present experience, powers and needs, and in such a way that its relationship to what is already significant to him, is fully appreciated. Subject matter is not first selected and then made interesting. This results in artificial devices being employed to compel attention. But where the intrinsic relationship is made, attention flows naturally.

True effort results when the end or purpose of one's activities is perfectly clear and his energy is directed into reflective judgment rather than into a blind thoughtless struggle. There can be no antagonism between these agencies of mental development. When the end is not direct but remote many additional difficulties arise and the accompanying activity is longer in realization of this end, requiring more thought and a greater need for effort. Effort

[1] Horne: "Philosophy of Education," p. 199.

becomes associated with this increased depth and scope of thinking in surmounting difficulties in the attainment of an end, of which the individual is fully conscious.

Motive then is "the name for the end or aim in respect to its hold on action, its power to move."[1] This end is the child's own and carries him on to possess the means of realizing it, and these will be in proportion to the identification of the end with some activity of the self. There can be no such identification unless the subject-matter of the curriculum is vitally connected with the child's present tendencies and activities.

When school work is of such a character that each child is bent upon the realization of some end or ideal, in other words when work becomes definitely purposeful in his life, the problem of moral training, if such there be, is happily solved. Right moral attitudes flourish when the child's activities and his self are thoroughly identified.

If school life is to develop the child so that his childhood will be rich and meaningful, there should be an organic motivation of the whole course as well as each subject. Sometimes a single event will provide real motives for practically all the school subjects. An instance of this is the School Fair. Not only does the preparation but also the events following furnish opportunities for developing the pupils' capacities in composition, art, arithmetic, etc.

With arithmetic in mind more particularly, one sees many sources for motivation arising out of the work in school agriculture, manual training and domestic science. The other subjects are of course not without their problems requiring solution by means of arithmetic. In no case need these sources be forced and unduly artificial.

Another source is the children's games and various school activities. Number games have already been spoken of. But the larger field of school games and activities provide many real problems to be worked out in class. The laying out of a baseball diamond or tennis court are but two instances. The various scores of the school teams and their percentage standings are other cases in point. A third source

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[1] Dewey: "Interest and Effort," p. 60.

is provided in the out-of-school and home experiences of the children, and these suggest themselves readily.

The following case well illustrates the point.[1] A certain town was supplied with a standpipe around which the school children often gathered. One day the teacher asked the seventh grade class if they had ever wondered about anything that perhaps some calculation could solve. Some of the class said they had often wondered how much water there was in the standpipe, but did not see how they could find out. This problem then became the subject of much investigation and after several days certain pupils came with a method of solution which they had discovered for themselves by employing a round stick of wood split into many pieces and piled so as to fit like wedges.

While doubtless thoughtful teachers have always given meaning and definiteness to their work along the lines indicated, yet the idea of motivation has not caught sufficiently to be the rule rather than the exception. It was behind such schools as the Dewey in Chicago. It is the dominant note in such schools as Mrs. Johnson's of Fairhope, Professor Meriam's,[1] the Francis Parker School, the elementary schools connected with Teachers College, Columbia University, and we hope many others. From these an influence is slowly radiating in ever increasing circles.

RELATION OF ARITHMETIC TO GEOMETRY AND ALGEBRA

This section must necessarily be curtailed, otherwise the discussion may easily develop into the question of secondary school mathematics, which is a big field in itself, and is not intended to be entered upon in this thesis.

In Chapter II, on Aims and Values, we purposely employed at times the broader term "mathematics" so that the values of geometry and algebra need not be repeated here. Writers upon the subject of mathematics have maintained that each of the branches of mathematics is characterized by

[1] Scott: "Social Education," p. 169
[2] Dewey: "Schools of Tomorrow."

requiring special and distinct mental capacity for attack.[1] This will not surprise the reader when he recalls the experimental work proving the complexity of arithmetical ability, referred to earlier. Most authorities agree too that school mathematics and the "higher" mathematics are two distinct fields. The comparative distinctness of the three branches of mathematics is an argument against their indiscriminate fusion in the elementary school as some would advocate.

The order of appearance of the subjects in the history of the race was, arithmetic, geometry, algebra. Evidence is not lacking to suggest that arithmetic and geometry were almost, if not, coeval, while experiments have shown that children's idea of form is very early in appearance.[2] The old manuscript extant treats of all three.[3]

We cannot here go into a detailed development of the subject of geometry and algebra. For geometry it must suffice to say that it had its origin in land-surveying. There are very early evidences of geometry among the Babylonians and Egyptians. The papyrus of Ahmes contains considerable work on mensuration. The antiquity of the Pyramids is proof that the Egyptians must have been well up in the science of geometry prior of 3000 B.C. We have already seen the keen interest the Greeks took in the science of space and the high perfection which geometry attained resulted in its being treated as a mature study and was carried on in the highest schools only. This will account for its failure to appear in the more elementary school until quite recent times. For instance it did not make any appearance in the secondary schools of Europe before 1600.

The first definite record we have of algebra is the famous manuscript of Ahmes. Here appear many crude mathematical symbols, applications of the linear equations and series. The first equation appeared in the form (using our symbols and letters) $\frac{x}{7} - x = 19$. In the Golden Age of Greece algebra was but a phase of geometry and not until the time

[1] Judd: "Psychology of High School Subjects;" also writers such as Smith, Young, Schultz, Rogers and others.

[2] Rusk: "Introduction to Experimental Education."

[3] Ahmes.

of Diophantes, 275 A.D., was any attempt made to have algebra stand upon its own legs. Later the Hindus gave almost exclusive attention to algebra. It is of more than historic interest to note that the new arithmetic, as revolutionized by Hindu numeration, and algebra, made practically concurrent appearances into modern Europe. Both sciences, however, took second place in importance and difficulty, in the universities, to geometry, which meant that both should be taught prior to geometry, although this order for algebra and geometry does not represent true intellectual sequence.

Until late years on this continent, the general plan has been to limit the eight grades of the public school to work in arithmetic (including mensuration), and leave geometry and algebra to the high school. It was customary too, to have an additional year or two on arithmetic in the high school as well. The Committee of Fifteen proposed cutting off arithmetic at the end of Grade VI and devoting the seventh and eighth grades to algebra with five periods a week. This suggestion was not original with the members of the committee, who had before them the practice of European schools where both algebra and geometry received attention much earlier than here. It is noteworthy that the Committee took the trouble to define the course in algebra otherwise than the prevailing course in the early classes of the high school.

The omission of this very point raised by the Committee accounts for the failure of the first attempts to introduce algebra into the elementary schools. The attempts simply amounted to transferring back into the grades, the class of work formerly done in the early years of the high school. This form of algebra lacked contact with the pupils, was too abstract, and bore little relation to arithmetic. Later a real effort was made to correlate the two, the solution of arithmetical problems by algebraic methods forming an easy transition. This did not mean, necessarily, the introduction of any text in algebra, though of late years improved texts have made their appearance and been adopted, especially in the eighth grade.

We have never taken to the idea of introducing demonstrative geometry below the high school. Many systems have met the demand for geometrical teaching by an extension

of the work in drawing and improved methods of teaching mensuration. All have recognized the value of constructive exercises beginning with the kindergarten and emphasized in the primary grades. Other systems have adopted experimental, practical and observational geometry in the last year or two of the eight grades. There is room for improvement, however, in the elementary texts used in this subject, looking to a closer connection and application with the interests of school children.

Keeping in mind the danger of transgressing into the field of secondary school mathematics, reference should be made to recent experimental work to ascertain the relation between school mathematics and higher mathematics, the character of mathematical ability, the processes involved in mathematical thinking, etc. While many investigations have been made, little real definite information has been furnished as to the nature of mathematics and as to the relation between mathematical and other abilities. The studies are all very suggestive, however.

A more recent study, and the resultant tests formulated to ascertain mathematical ability and correlation between it and other abilities, is bound to have far-reaching results, not only in the field of high school mathematics but also in the senior grades of the public school. I can do nothing more than quote slightly from the author's introductory chapter. "Its purpose is to make an analysis of the abilities involved in high school mathematics, to determine their efficiency and status, their inter-relations and also their connection with certain other forms of mental capacity. Primarily it is directed to discover dynamic and quantitative relations between mathematical abilities rather than to show how we think in mathematics from the standpoint of analytic or structural psychology. . . ."[1]

MEASUREMENTS IN ARITHMETIC.

The age we are entering demands, in every department, that results be measurable. The school is no exception and the teacher is beginning to realize that teaching is not good

[1] Rogers: "Tests of Mathematical Ability and Their Prognostic Value."

unless it produces results that can be measured quantitatively, and by tests not of his own making; and no longer is the pupil's performance left to the single judgment of his teacher.

Perhaps the heaviest score against the traditional examination system, is the inaccuracy of the teachers' marks. Anyone who has had to do with reading of "departmental" examinations, knows how teachers vary in their estimate of answers, even where the greatest care has been taken to coach the examiners looking towards uniformity. Unless the very closest supervision is maintained, many pupils will pass who should not and many will fail of promotion who are quite deserving. The subject of unreliability of teachers' marks has been investigated by several,[1] and all agree that teachers' marks are in general inaccurate measures of pupils' abilities.

An interesting investigation by Starch and Elliott, reported in School Review, Vol. II, was made in grading mathematics. A copy of an answer paper in geometry was sent to 116 persons to mark on the basis of 100. Forty-seven of these persons would pass the candidate, while sixty-nine would pluck him. Two marked the paper above 90; one below 30; etc. This well illustrates the unreliability of teachers in the matter of marking pupils' work. All investigations bear out the conclusion that the candidate would do well to select his own examiner. But even here there is a danger, for teachers are not always consistent.

Rice was perhaps the pioneer in the matter of standard tests. In 1897 he rather startled the school world with his conclusions in the question of spelling. Five years later he made another experiment with 6,000 school children, this time in arithmetic. Since then progress in the field of standard tests has been rapid. We are far from perfection yet and each year sees some improvement in this important field. Many are strongly prejudiced against such scales, being too closely wedded to the old examination tests. Others again have been carried away by their novelty but do not appreciate their significance. These do the cause no good.

In proposing to employ a test one must know what he is about to measure, just what is the outcome of instruction in

[1] For instance, Carter: "Elementary School Teacher," November, 1911

the branch of study under consideration. In arithmetic there are several desirable immediate ends. Abilities in the fundamental operations and abilities in reasoning are the ones for which standard tests have been chiefly prepared. Arithmetic we found, is a complex of abilities, there being as many abilities as there are types of examples, all more or less independent, some absolutely so. A thorough test would require an examination of every type of example. Fortunately, however, some types include in them abilities used in lesser types. For instance, in four-column addition is involved all the elements in two and three column addition, so that if a subject pass the test in four column addition, it can be assumed that he has the ability covered by the other additions.

The investigation of Stone in 1908 of Grade VI pupils in 26 school systems exhibited several very important results, some of which have already been referred to. What impressed him most was the wide variation in fundamentals among the various systems; and further the fact that the order of merit for such performances did not necessarily agree with the order of merit for reasoning. He also found that the correlations of reasoning with the average of the fundamentals (in case of whole systems) was quite low, especially in case of addition. In case of subtraction the coefficient was the highest. In individual scores this held too, except that division showed the highest correlation. This would means that division is most like reasoning, addition the least. Addition ability is no guarantee of ability in the other fundamentals.

We have before mentioned the findings as regards complexity of arithmetical ability and the relation between time spent and accomplishment. All these results indicated that teachers for the most part were working in the dark and had no real criterion to guide them.

Stone's results inspired others who have worked upon standard tests and these have been published.[1]

The first perhaps was Courtis. His experiments verified many of Stone's conclusions and had the advantage of cover-

[1] Gray: "Elementary School Teacher," September, 1916, gives a good descriptive list of standard tests.

ing all the school grades whereas those of Stone applied to Grade VI only. Courtis found the same complexity of abilities as did Stone and found further that as the pupils and classes go up the grades they do not advance uniformly but show great differences in rate and quality of improvement. The tests as finally evolved by Courtis assist the teacher to compare the efficiency of individuals and classes with established norms.

The Courtis Tests in the simple operations consist of four tests, 24 questions in each, the examples of each test being approximately of the same difficulty. The times allowed are 8, 4, 6 and 8 minutes respectively. The directions given the pupils are as follows: You will be given 8 minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

Following are samples from the various tests:

Addition:

927	297	136
379	925	340
756	473	988
837	983	386
924	315	353
110	661	904
854	794	547
965	177	192
344	124	439

Subtraction:

107795491	75088824	91500053
77197029	57406394	19901563

Multiplication:

8246	3597	5739
29	73	85

Division:

25)6775	94)85352	37)9990

Results are scored on the number attempted, no marks being given for examples partly right or incomplete.

A glance at the tests, especially those in addition, may incline one to remark that these are not typical of ordinary addition problems. A consideration of the following will answer this objection. The Courtis Tests are to measure "end products"; that is, they are designed to measure the most complex form in which a given skill is found. Now it is more or less conventional what type of ability should be selected to represent the end product. Courtis adopted the principle that in any operation the units selected shall be the smallest that cover all essential elements. For the different operations these elements are as follows:

Addition:

1. Knowledge of combinations.
2. Bridging the tens.
3. Carrying.
4. Attention span.
5. Fatigue.

Subtraction:

1. Knowledge of combinations.
2. Borrowing.
3. Fatigue.

Multiplication:

1. Knowledge of combinations.
2. Place value.
3. Carrying.
4. Addition.
5. Fatigue.

Division:

1. Knowledge of combinations.
2. Place value.
3. Estimation of quotient.
4. Multiplication.
5. Subtraction.
6. Fatigue.

Hence the figures in the various examples were selected with the greatest care according to a systematic plan. Care is taken in all of the tests to cover for each operation every factor mentioned above, and sufficient material is provided to keep the brightest pupil busy for at least four minutes. For in four minutes the average child will reveal any marked tendency to make errors due to fatigue.

Mr. Courtis has also arranged Standard Supervisory Tests for the assistance of supervisors and inspectors. Test A, Form 1, covers the four operations without carrying, test B, with carrying. As these tests are short, I give them here complete.

Test A, Form 1

Instructions: Work as many of these examples as you can in the time allowed. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

Add

5	1	3	7	4	4
1	5	2	2	9	1
3	4	4	8	8	3
9	7	9	1	5	6
4	4	5	5	1	5
8	1	4	9	4	7
1	7	3	5	2	9

Subtract

140	122	161	172
73	77	68	84
111	192	120	183
72	98	64	96
114	137	94	132
15	57	70	53

Multiply

41	54	21	43	61	52
17	22	16	23	54	34

Divide

41)1558 53)636 61)3477 42(882 51)1479

Name, etc.

Test B, Form 1

Same instructions.

Add

54	41	44	42	60	69	64	38
43	77	96	87	45	33	85	49
55	89	88	76	99	67	66	95
79	44	57	83	38	96	83	57
95	89	74	68	76	63	87	98

Divide

42)3528 56)4088 64)4352 65)4680

Name, etc.

The necessary class record sheets accompany the tests and the supervisor is thus able to keep records for his various schools and classes.

Standard Practice Tests are now being employed to automatically diagnose each pupil and give him the very practice he needs. Each child is supplied with a student's Record Book and Practice Pad and an envelope containing 48 lesson cards. Each lesson card consists of a number of examples of one type, the types being so chosen that the range is from the simplest examples to the most difficult a child in the grades is called upon to solve. Further, the mastery of the examples on each card insures the mastery of some one of the many component elements that enter into skill in the four operations. A lesson card, practice side up, is placed under the topmost tissue-paper sheet of the pad. The examples are seen through the paper quite distinctly, and the work is done of course directly upon the tissue paper, so that at the end of the practice period the card may be taken out. The answers are on the back of the card and the pupil may compare his answers with these. Only the supply of tissue sheets or pad needs to be renewed. The scores made each day are recorded, graphs drawn, and the child is able to see his improvement, if any.

These practice tests are of value in the ungraded rural school as well as in the big city school system. The same lessons are given for all Grades IV to VIII, the only difference being in time allowance. Thus all the pupils may be instructed at the same time, each getting the practice he needs most.

The Cleveland-Survey Tests, designed by Professor Judd, are designed to diagnose the various arithmetical processes. There are a series of 15 tests numbered from A to O. Tests A, E, J and M are addition; B and F, subtraction; C, G and L, multiplication; D, I, K and N, division; H and O, fractions. The time allowed varies from thirty seconds to three minutes. The examples of each test are approximately equal in difficulty.

Set A contains thirteen examples like $\frac{4}{2}$ $\frac{7}{1}$ etc., Set B, ten examples like $\frac{7}{3}$ $\frac{13}{8}$ etc., set C, ten examples like $\frac{7}{4}$ etc., set D, seven examples like 3)9, 4)32, etc. From here on, the operations appear in a more elaborate form. For

instance there is single-column addition of five figures, single-column addition of thirteen figures, four-column addition of five rows. There is similar increasing complexity in the other operations.

Besides their usefulness for diagnosing purposes, the Cleveland Tests of course measure the efficiency of the class in the operations embraced by the tests. Standard norms have been established, and anyone wishing to do so may compare his school or entire system with these established norms.

It will occur to the reader that the test is not complete enough, particularly in the field of common fractions. Their real strength is in the detailed measurements they furnish of the abilities covered.

The Woody Scales[1] consist of two series, A and B, the latter being an abbreviation of the former and used when the time is not available for the complete series. Series A contains four sets, one for each of the fundamentals. The addition scale, for instance, contains 38 questions involving simple combinations, column addition, addition of dollars and cents, decimals, fractions and denominate numbers. In each set the questions are supposed to be in ascending order of difficulty. Time allowed for each set is twenty minutes.

It should be noted that a test and a scale are not the same. In the tests described above, the questions were of equal difficulty in any one set of operations. In Woody's Scales they are of increasing difficulty, and the pupil's performance is a statement of the particular examples he has done correctly. To secure a class score cognizance is taken of the example which is done correctly by one-half the class. It will be seen that "ability" as used in the Courtis Tests, for instance, and these Scales, is not quite the same thing. Then, too, no particular account is taken of speed, as most pupils find the twenty minutes ample.

The scales, then, are intended as measures of achievements of a class or school in the four fundamentals, in all the variety of ways that the simple rules may be applied. The scales also furnish an excellent diagnosis of the quality

[1] Appendix F.

of the class work, an illustration of which is given below.[1] The author of the scales has worked out a valuation for each of the questions, and these valuations are to be used in calculating the achievement of the class. The point in scoring is, as intimated a moment ago, to find how difficult a problem must be in order to be accurately solved by 50 per cent. of the class. The scales were worked out on this principle.

The Monroe Standard Reasoning Tests are three in number. Test 1 is to be used in Grades IV and V; Test 2, in Grades VI and VII; Test 3, in Grade VIII. Each test contains 15 problems, and each problem is marked on the basis of principle and correct answer. The relative values of these vary in different questions.

The following two questions from Test 1 will give one an idea of the nature of the paper. Question 1: Mr. Black received $2 a yard for broadcloth. He sold 78 yards. How much did he receive? There are two points given for principle and two for correct answer. Question 7 reads: A car contains 72,060 lbs. of wheat. How much is it worth at 87 cents a bushel? Here principle gets four, correct answer three.

The general directions are printed on the class Record Sheet. They are as follows:

1. In order that a solution may be counted as correct in principle, the correct operation must be performed upon the correct numbers. For example, if a problem requires division, the right number must be used as a divisor.

2. A solution is counted as correct in principle when the operation performed is based upon the true relations of the numbers used.

3. The fact that a pupil does not use the shortest method is not to be counted against him. Our thinking is psychological. It is only necessary that each operation in the solution be based upon the true relations of the quantities.

4. Errors in denominate numbers (such as the number of months in a year, number of pounds in a ton, number of square feet in a square yard, etc.) do not affect the correctness of the principle.

[1] Page 144.

5. No credit is given for a problem partially correct in principle. In a two-step problem, no credit is given unless the pupil at least has definitely indicated the last operation.

6. Answers are counted as correct only when the answer is numerically correct. If it contains a fraction it must be reduced to its lowest terms. Answers need not be labelled; that is, if the answer is square feet it is not necessary that it be labelled as such.

7. If a solution is incorrect in principle, the answer is also marked incorrect.

8. If a pupil solves a problem correctly and then continues with additional operations which are not called for, his solution becomes incorrect in both principle and answer.

It will be seen that considerable care must be taken in scoring, and places may occur where reasonable doubt may be raised, especially where principle is involved. Occurrences of this kind appeared in our experiments, some of which are noted in the succeeding chapter.

The writer did considerable experimental work with standard tests, especially with Courtis Tests, Cleveland-Survey Tests, Woody Scales and Monroe's Reasoning Tests. An account of these is reserved for the next chapter.

CHAPTER VI.

SOME EXPERIMENTAL WORK WITH STANDARD TESTS.

In order to ascertain the value of Standard Tests and the conditions under which they might be employed, a number of experiments were undertaken employing children of certain selected public schools, the Collegiate and the Normal School. The tests made use of were the Courtis-Standard Research Tests, Series B, Forms 1 and 2; Woody Scales, Series A; the Cleveland Arithmetic Exercises, and Monroe's Reasoning Tests. In all, between 1,500 and 1,800 tests were made. Unfortunately, owing to an epidemic of mild influenza, pupils and teachers were very irregular in attendance and many of the rooms were reduced greatly in numbers. In several instances substitute teachers were in charge of rooms, with the result that it was impossible sometimes to secure the information desired concerning the pupils. However, some interesting results followed, and the way was opened for further experimental work at a more favourable season.

There was no intention of making a survey of the city's schools. In fact, the teachers were assured that comparisons between schools would be reduced to a minimum. This will account for the absence from this chapter of comparative data which would in any way reflect on the standing of various rooms and schools. Much interesting material revealed by the tests is necessarily eliminated.

As the Courtis Tests are the best standardized, considerable interest centred around their employment. Table I contains Standard Median Scores, using the Courtis Tests, Series B (R=rate; A=accuracy).

TABLE I

Grade	Addition R	Addition A	Subtraction R	Subtraction A	Multiplication R	Multiplication A	Division R	Division A
IV.								
General	7.4	64	7.4	80	6.2	67	4.6	57
Courtis	6	100	7	100	6	100	4	100
Boston	8	70	7	80	6	60	4	60
V.								
General	8.6	70	9.0	83	7.5	75	6.1	77
Courtis	8	100	9	100	8	100	6	100
Boston	9	70	9	80	7	70	6	70
VI.								
General	9.8	73	10.3	85	9.1	78	8.2	87
Courtis	10	100	11	100	9	100	8	100
Boston	10	70	10	90	9	80	8	80
VII.								
General	10.9	75	11.6	86	10.2	80	9.6	90
Courtis	11	100	12	100	10	100	10	100
Boston	11	80	11	90	10	80	10	90
VIII.								
General	11.6	76	12.9	87	11.5	81	10.7	91
Courtis	12	100	13	100	11	100	11	100
Boston	12	80	12	90	11	80	11	90

Table II will indicate the abilities of various groups of persons, using the Courtis Tests.

TABLE II

Group	No. in Group	Addition R	Addition A	Subtraction R	Subtraction A	Multiplication R	Multiplication A	Division R	Division A
A	38	8.5	61	9	80	8	61	7.6	66
B	35	10	59.2	12.6	71.6	10.6	59.9	8.7	75
C	27	10	55	12	78	9.4	72	9.6	84
D	28	13	76	15	84	13	72	14	90
E	55	15.8	80.8	16.7	82	14.1	77	16.8	94.7
F	23	12.7	73.5	16.4	90	13.5	79	14.5	91.4
G	13	9	65.2	7	70.6	5.8	54.7	3.5	50
H	6	9.5	66.7	15.2	80.2	10.3	71	8.5	82.4
I	14	11.5	75.8	12	87.4	7.3	63.7	5.4	75
J	6	14.2	86	16.7	83	15	85.6	14.7	94.3
K	6	19.7	83.9	22	88.6	20	88.3	19	96

A. A Grade VIII class, 5 months before Entrance Examination.

B. A similar Grade VIII.

C. A first year High School Class.

D. A class of teachers in training, few with teaching experience; each held a second class diploma.

E. A somewhat similar class, each having had at least one year's experience.

F. A similar class to D, but each member held a first class diploma.

G. A night class (mixed) of young people.

H. A somewhat more advanced class.

I. A night class of adults of non-English extraction.

J. A group of graduates in Arts.

K. A group of grade teachers in the public school.

In order to ascertain the reliability of a single test to measure the class score, two performances on two successive days were secured from one of our classes. The class consisted of some 35 young men and women, teacher-in-training, few of whom had ever taught. Most of them had graduated from the High School some six months previously.

The four operations were done at one sitting, with intervals of one to two minutes. At the conclusion of each performance every student declared he was "fagged out"; many had headaches. All of which would indicate that the Courtis Test should be given in at least two sittings.

The results are given below. Only those students' results were included who were present on both occasions, 28 in all. Many were absent one or both times on account of illness. Each student's record is set out in order that individual variations may be seen. Table III gives the attempts; Table IV the accuracies worked out on a percentage basis.

TABLE III

Attempts.

Pupil	Addition		Subtraction		Multiplication		Division	
	I	II	I	II	I	II	I	II
A	8	10	10	13	11	13	11	12
B	12	17	11	12	9	11	12	15
C	17	17	15	18	13	15	15	11
D	12	13	12	14	10	12	15	15
E	18	18	14	17	14	17	12	16
F	15	17	17	21	14	16	16	18
G	15	18	20	21	16	18	17	18
H	14	16	24	24	18	21	14	17
I	11	9	10	10	11	12	13	12
J	8	11	12	14	10	12	22	19
K	16	19	24	24	18	17	22	19
L	22	23	24	24	20	21	24	24
M	13	14	13	13	9	10	13	16
N	17	18	17	19	14	15	14	19
O	7	6	11	10	9	11	9	8
P	15	16	24	24	16	17	15	17
Q	20	21	20	24	18	18	22	22
R	15	16	18	19	15	16	15	18
S	9	9	9	12	8	10	6	15
T	10	10	13	12	7	11	7	10
U	16	20	17	21	18	19	22	24
V	10	10	9	11	10	13	14	13
W	10	11	14	17	12	15	11	16
X	14	14	17	19	13	15	17	18
Y	18	12	13	11	12	14	9	11
Z	17	18	17	23	15	18	12	15
AA	8	9	10	11	8	10	10	10
AB	7	8	12	14	8	9	11	14

TABLE IV

Rights.

Pupils	Addition		Subtraction		Multiplication		Division	
	I	II	I	II	I	II	I	II
A	100	90	70	85	64	85	82	92
B	92	92	82	100	78	91	85	100
C	77	88	87	94	77	73	93	100
D	100	100	92	100	100	100	100	100
E	33	67	64	48	54	59	75	89
F	100	82	94	95	79	94	87	83
G	53	72	55	67	44	61	82	78
H	79	75	83	88	56	81	86	100
I	64	78	90	90	82	83	100	100
J	100	100	92	93	90	83	100	100
K	94	95	92	75	94	94	95	100
L	100	96	92	96	90	96	83	96
M	77	97	77	100	56	90	85	75
N	71	72	94	84	79	60	100	100
O	100	50	64	60	44	64	100	87
P	80	81	67	92	63	82	100	94
Q	70	95	100	96	78	89	86	100
R	80	81	100	95	80	100	87	94
S	56	89	89	100	63	90	100	100
T	90	70	54	75	57	55	86	100
U	69	75	88	86	89	74	100	100
V	70	80	67	64	50	54	93	93
W	90	73	85	82	42	60	90	88
X	64	79	76	68	69	67	82	100
Y	78	75	100	91	83	71	56	82
Z	94	83	94	100	87	100	100	100
AA	0	44	100	100	50	80	90	86
AB	0	50	100	93	75	78	91	93

The results in attempts may be summarized thus:

	Addition	Subtraction	Multiplication	Division
1st attempt	374	427	356	393
2nd attempt	400	472	406	416
Difference	26	45	50	23

These differences or gains in attempts are spread over 28 pupils.

The number of rights may be shown in a like manner:

1st attempt	283	359	256	355
2nd attempt	319	408	322	442
Gains	36	49	66	87

The improvement in accuracy for the class:

	per cent.	per cent.	per cent.	per cent.
1st attempt	76	84	72	90
2nd attempt	80	86	79	94
Gain	4	2	7	4
Average of two trials	78	85	75.5	92

The averages for the two trials for attempts is shown thus:

Average	14	16	14	14.5
1st trial	13	15	13	14

To further investigate the reliability of a single trial, the addition and subtraction sets were given on two occasions with a day's interval to a Grade IV class. Tables V and VI contain the results. (A1=attempts first trial; R1=rights first trial; A2=attempts second trial, etc.)

TABLE V

Addition.

Pupil	A1	R1	A2	R2	Pupil	A1	R1	A2	R2
A	10	8	8	5	N	5	3	5	1
B	8	6	7	7	O	6	5	5	3
C	8	2	8	7	P	7	2	6	6
D	7	3	7	3	Q	10	9	10	9
E	8	4	8	5	R	6	5	7	6
F	4	3	3	0	S	8	0	8	1
G	8	6	7	3	T	3	2	4	3
H	7	1	8	1	U	8	1	7	3
I	6	6	7	7	V	4	1	4	0
J	7	1	6	2	W	9	5	9	5
K	6	0	6	3	X	7	6	8	7
L	2	1	8	4	Y	8	4	7	7
M	7	5	9	7	Z	8	5	8	8
					Total	177	94	180	113

TABLE VI

Subtraction.

Pupil	A1	R1	A2	R2	Pupil	A1	R1	A2	R2
A	6	6	8	7	N	1	0	3	0
B	3	2	7	7	O	4	0	4	1
C	3	0	9	1	P	7	5	9	7
D	5	3	9	6	Q	2	1	7	4
E	6	5	12	12	R	3	0	5	2
F	3	3	4	1	S	4	1	7	5
G	4	3	8	5	T	6	1	11	5
H	6	1	8	2	U	5	3	10	8
I	4	2	7	3	V	4	4	6	3
J	7	5	9	6	W	6	6	9	8
K	4	1	6	3	X	7	5	15	11
L	3	3	8	8	Y	6	5	11	8
M	3	2	6	4					
					Total	112	67	198	127

The class results may be summarized thus:

	Addition		Subtraction	
	Rate	Accuracy	Rate	Accuracy
1st trial	6.8	53.1	4.5	59.8
2nd trial	6.9	62.7	7.9	64.1
Average	6.85	57.9	6.2	62
Increase over 1st trial	nil	4.8	1.7	2.2

A somewhat different experiment was undertaken in another Grade IV. Here the addition test was given on each of four successive mornings. On each occasion the pupils began work at different points in the set to avoid any possibility of memorization of results.

Table VII shows the results. Very little change is noted in the rate from day to day. In accuracy, however, the average of four trials is much higher than the first attempt. The increase is greater than one should expect, and an inquiry was made as to any probable cause. The teacher in charge of the class admitted that after the first trial, and during my absence, she had advised the class to work less rapidly and exercise more care. It would appear that the better showing in "rights" in Trials II, III and IV may be partly accounted for on this ground:

TABLE VII

Pupil	Attempts I	II	III	IV	Av.	D	Rights I	II	III	IV	Av.	D
A	6	6	7	7	6.50	.50	3	5	5	7	5.00	2.00
B	9	6	8	10	8.25	−.75	5	5	8	9	6.75	1.75
C	7	7	7	5	6.50	−.50	4	6	4	5	4.75	.75
D	5	6	4	6	5.25	.25	2	3	2	3	2.50	.50
E	9	7	6	8	7.50	−1.50	1	6	4	4	3.75	2.75
F	10	4	12	10	9.00	−1.00	5	1	8	7	5.25	.25
G	9	7	7	9	8.00	−1.00	0	4	3	4	2.75	2.75
H	7	7	6	7	6.75	−.25	3	3	3	3	3.00	.00
I	11	11	10	12	11.00	.00	8	9	9	10	9.00	1.00
J	5	5	5	5	5.00	.00	3	5	4	4	4.00	1.00
K	5	6	5	7	5.75	.75	0	5	3	4	3.00	3.00
L	10	7	9	9	8.75	−1.25	7	4	7	8	6.50	−.50
M	6	7	6	7	6.50	.50	2	4	4	4	3.50	1.50
N	8	8	8	7	7.75	−.25	5	6	5	3	4.75	−.25
O	5	5	5	4	4.75	−.25	1	3	4	0	2.00	1.00
P	7	8	8	9	8.00	1.00	4	7	8	6	6.25	2.25
Q	8	8	8	8	8.00	.00	8	7	8	8	7.75	−.25
R	3	3	4	4	3.50	.50	0	0	1	1	0.50	.50
S	10	9	8	10	9.25	−.75	8	9	7	9	8.25	−.25
T	4	5	5	6	5.00	1.00	2	3	5	6	4.00	2.00
U	9	9	9	7	8.50	−.50	6	6	8	3	5.75	.25
Totals	153	141	147	157	149.50	−3.50	77	101	110	108	99.00	22.00

Rate: 7.3, 6.7, 7.0, 7.5, 7.1, −.2 Accuracy 50.3, 71.6, 74.8, 68.8, 66.2, 15.9

(The Roman numerals represent the various trials: Av. = average of four trials; D = deviation from the average of the first trial.)

Summing up the findings upon the reliability of a single test, one is inclined to favour the average of several trials not too distantly separated. However, except in Table VII, the average of the trials and the first trial are not in great disagreement, so that the first trial will represent fairly approximately what the class can do. Undoubtedly pupils should improve a little with each trial, as each performance is practice for the next.

Table VII contains a suggestion that in the first trial the pupils did not really "rise" to the occasion. The appeal of the class teacher undoubtedly had results in the successive trials.

The influence of fatigue was shown by the following experiment. A Grade VIII was selected and given the Courtis Tests at two sittings. During the next ten days the Woody Scales were given mostly on alternate days. The teacher of this room, becoming interested in tests,

secured Standard Writing Tests, and these were given during the same period. As a result the room was fairly "charged" with tests, and the pupils were all on their mettle. No particular work in arithmetic, other than the usual class work, was given.

The writer then gave a second test, using the Courtis Tests, but this time in one sitting. The results are contained in Tables VIII and XI. Individual scores are given so that the variations may be studied by anyone interested.

TABLE VIII

ATTEMPTS.

Pupil	Addition			Subtraction			Multiplication			Division		
	1st	2nd	Inc	1st	2nd	Inc	1st	2nd	Inc	1st	2nd	Inc
A	14	11	−3	12	12	0	10	10	0	8	9	1
B	11	9	−2	9	11	2	10	11	1	8	7	−1
C	9	9	0	14	15	1	14	13	−1	10	14	4
D	7	6	−1	9	11	2	8	8	0	6	4	−2
E	8	9	1	7	13	6	7	9	2	7	6	−1
F	6	7	1	7	7	0	7	8	1	7	7	0
G	13	12	−1	13	14	1	11	11	0	13	9	−4
H	9	10	1	9	10	1	10	10	0	6	4	−2
I	8	7	−1	8	9	1	9	12	3	8	8	0
J	9	10	1	9	10	1	6	7	1	4	4	0
K	11	11	0	11	11	0	9	8	−1	9	7	−2
L	8	9	1	11	11	0	9	9	0	8	10	2
M	7	8	1	9	10	1	6	5	−1	2	4	2
N	16	16	0	11	13	2	9	8	−1	12	8	−4
O	8	8	0	8	8	0	7	8	1	8	6	−2
P	12	14	2	7	10	3	9	8	−1	8	10	2
Q	6	11	5	9	13	4	11	10	−1	10	8	−2
R	7	8	1	10	10	0	7	8	1	7	6	−1
S	11	13	2	17	17	0	11	12	1	9	7	−2
T	6	8	2	7	9	2	6	7	1	8	6	−2
U	6	7	1	7	5	−2	4	3	−1	8	7	−1
V	8	9	1	10	13	3	9	10	1	10	11	1
W	11	9	−2	12	12	0	8	8	0	12	11	−1
X	6	7	1	6	7	1	5	5	0	3	4	1
Y	7	8	1	7	6	−1	6	4	−2	5	4	−1
Z	9	9	0	11	14	3	11	10	−1	7	8	1
Aa	9	8	−1	7	9	2	9	12	3	10	11	1
Bb	12	11	−1	10	13	3	8	8	0	8	8	0
Cc	9	10	1	9	8	−1	7	7	0	4	3	−1
Dd	6	6	0	8	7	−1	6	6	0	6	5	−1
Ee	6	8	2	6	11	5	7	7	0	7	9	2
Ff	1	3	−1	4	3	−1	4	2	−2	2	1	−1
Gg	6	7	1	9	7	−2	7	7	0	6	3	−3
Hh	6	6	0	9	9	0	6	6	0	8	8	0
Ii	9	12	3	9	11	2	10	10	0	13	12	−1
Jj	8	10	2	12	14	2	10	9	−1	8	6	−2

TABLE IX

Rights.

Pupil	Addition			Subtraction			Multiplicatoin			Division		
	1st	2nd	Inc	1st	2nd	Inc	1st	2nd	Inc	1st	2nd	Inc
A	7	9	2	11	12	1	9	9	0	7	8	1
B	10	9	−1	7	10	3	10	9	−1	8	7	−1
C	5	5	0	12	15	3	10	9	−1	10	14	4
D	4	4	0	8	10	2	6	7	1	5	2	−3
E	6	5	−1	4	11	7	6	6	0	7	5	−2
F	3	6	3	7	6	−1	7	8	1	7	5	−2
G	6	11	5	11	13	2	9	11	2	11	9	−2
H	5	3	−2	8	10	2	6	6	0	4	1	−3
I	6	7	1	6	9	3	8	8	0	8	8	0
J	8	9	1	7	10	3	1	3	2	3	3	0
K	10	10	0	10	9	−1	8	8	0	6	6	0
L	5	7	2	6	7	1	8	6	−2	7	10	3
M	1	7	6	6	9	3	4	1	−3	1	4	3
N	12	13	1	9	10	1	8	3	−5	11	5	−6
O	6	3	−3	4	7	3	3	3	0	6	4	−2
P	10	10	0	6	9	3	9	8	−1	8	10	2
Q	3	6	3	6	9	3	10	9	−1	9	11	2
R	5	4	−1	9	9	0	4	5	1	7	5	−2
S	8	11	3	15	15	0	8	7	−1	8	7	−1
T	3	6	3	4	7	3	5	4	−1	8	3	−5
U	4	4	0	5	5	0	2	2	0	6	7	1
V	2	7	5	9	13	4	4	9	5	10	11	1
W	6	9	3	8	11	3	8	5	−3	11	10	−1
X	4	4	0	6	6	0	4	5	1	2	3	1
Y	0	6	6	4	6	2	3	3	0	1	2	1
Z	1	6	5	6	9	3	7	12	5	7	10	−3
Aa	9	8	−1	11	13	2	9	10	1	6	7	1
Bb	6	6	0	8	12	4	6	5	−1	8	7	1
Cc	7	10	3	7	6	−1	4	4	0	3	0	−3
Dd	3	4	1	5	6	1	3	4	1	5	4	−1
Ee	2	5	3	4	6	2	4	2	−2	6	8	2
Ff	1	2	1	2	1	−1	1	0	−1	0	0	0
Gg	3	3	0	5	7	2	4	7	3	5	2	−3
Hh	1	5	4	7	9	2	5	3	−2	8	8	0
Ii	7	9	2	8	11	3	8	9	1	13	11	−2
Jj	6	9	3	12	13	1	9	7	−2	7	5	−2

One should naturally look for some improvement. This is indicated in the case of addition and subtraction; but

in multiplication, while there was a slight increase in the number of attempts, the rights showed a small decrease: while in division, decreases were registered in both attempts and rights. Table X summarizes the results. Only the pupils, 36 in number, who were present during all performances, were considered.

TABLE X

	Addition		Substraction		Multiplication		Division	
	A	R	A	R	A	R	A	R
First	308	185	333	263	293	220	275	236
Second	326	242	375	331	296	217	254	222
Increase	18	57	40	68	3	−3	−21	−7

There is no doubt that the strain of the single sitting is great and the results in multiplication and division will be vitiated considerably. In the writer's opinion, especially where comparisons are to be drawn, the test should be given in at least two sittings, and with young children possibly four.

The effect of practice is seen from Table XI. A Grade VII was given the division set. Then followed an interval of three days, during which the class was given daily drill on similar division questions and also home exercises. The rate increased from 8 to 11 and accuracy from 90 to 94. The increase in rate is particularly encouraging.

TABLE XI

Division.

Pupil	A1	A2	R1	R2	Pupil	A1	A2	R1	R2
A	8	7	6	4	L	7	4	7	9
B	8	7	8	6	M	7	10	6	8
C	5	6	5	6	N	7	10	7	9
D	9	13	9	11	O	9	12	9	10
E	8	11	8	11	P	8	16	4	16
F	6	9	5	8	Q	12	16	9	15
G	8	10	8	10	R	8	12	7	12
H	12	15	12	15	S	8	13	8	13
I	9	12	8	12	T	9	14	7	13
J	8	11	8	11	W	8	10	7	9
K	7	13	6	13	V	7	12	6	12

During the work the importance of absolute accuracy in keeping time was impressed more than once. To ascertain what an error of one minute would mean, the Courtis Tests, addition only, were given to one of our classes of 58 students. The students were instructed, upon time being called, to mark the question completed but to work on until stopped at the end of another minute. Records were kept for the addition. Below are the results:

Rate for 8 minutes	14.7	Accuracy	78.67%
Rate for 9 minutes	16.3	Accuracy	77.83%
Difference	1.6		−0.84

This table contains a surprising result. While attempts show a difference of 1.6, accuracy over the nine-minute period was actually lower than over the eight-minute period. This is contrary to what one should expect, if there is anything in the "warming up" feature of work, especially in a period of such short duration as to exclude the element of fatigue. Upon expressing surprise to the class, it was found that the great majority became flustered upon time being announced, and while attempts increased satisfactorily, the percentage of rights was not maintained.

The following day the experiment was repeated, this time using the division set. Somewhat similar results followed.

The addition test was given to another class under similar conditions. This class showed an increase in attempts, but an actual decrease in accuracy.

In a small interval of eight or nine minutes there should be little difference one way or the other in accuracy. Undoubtedly the large difference that occurred was due to emotional effects. Errors in timing, as a rule, will affect only the rate, and in making comparisons this is important.

In order to ascertain the effect of time of day when the test is given, the Courtis Series in Addition and Subtraction was given to a class of Grades III and IV at 9.20 A.M. and again at 3.40 P.M. the same day, the pupils on the second trial working from the end question in order to avoid memorization of results. Table XII gives results set out in full:

TABLE XII

Pupil	Addition				Subtraction			
	First Trial		Second Trial		First Trial		Second Trial	
	A	R	A	R	A	R	A	R
A	8	6	8	4	8	0	6	0
B	5	1	5	2	6	0	4	1
C	5	3	4	0	6	0	5	0
D	5	2	6	2	6	4	6	5
E	5	4	5	2	5	4	4	2
F	7	5	6	2	7	5	9	7
G	6	5	5	2	4	2	5	4
H	6	3	6	3	4	1	3	2
I	4	0	5	2	4	0	5	1
J	7	2	6	3	3	0	4	2
K	8	4	8	6	4	1	4	0
L	7	1	6	2	4	3	6	4
M	7	0	7	0	5	3	7	4
N	8	2	9	1	5	3	4	3
O	6	4	6	3	7	7	7	5
P	6	4	6	5	6	3	6	4
Q	7	5	7	6	5	3	6	4
R	8	5	7	3	7	5	8	7
Totals	115	59	112	48	96	44	99	55
	Accur. 51.3		Accur. 42.8		Accur. 45.8		Accur. 56.5	

The rate was slightly decreased in the case of addition and increased in subtraction. In accuracy the same relations held. It would appear that, while fatigue may have operated during the operation of addition, the pupils made a spurt and broke their record of the morning in subtraction. While a single experiment is not enough to arrive at conclusive results, yet there is contained a suggestion that pupils can overcome their apparent fatigue.

The Cleveland Tests were not employed to the same extent as were the Courtis. Their principal use arises out of their diagnostic value.

To ascertain particularly the use of the Cleveland Tests for class diagnosis, a Junior V class of 12 pupils was selected. The results for addition only are tabled below:

Addition.

Pupil Number	Number of misses in A	E	J	M
4	0	0	1	2
5	0	0	2	2
10	0	0	2	1
12	1	0	0	2
15	0	0	1	0
17	1	1	2	1
19	0	0	1	1
20	0	1	0	0
21	0	1	0	0
22	0	1	2	1
23	0	0	0	1
26	0	0	0	1

Recall for the moment that Set A contains simple combinations; E, single-column addition of 5 figures; J, single-column of 13 figures; M, four-column of 5 rows.

An examination of this table in conjunction with the answer papers revealed the following rather obvious conclusions: The errors in A are slips, though pupils 17 and 22 are weak in all forms of addition; pupils 4, 5, 10 and 19 are probably weak in carrying, for single column addition exhibited no errors, though J and M in conjunction might indicate fatigue. Pupils 12, 23 and 26 would appear weak in carrying, all having no errors in J, single column addition of 13 figures; pupil 15 is probably unable to hold partial sums for any considerable time; pupils 20 and 21 very likely made mere slips.

The above conclusions were afterwards fairly well substantiated by an oral examination of the pupils on similar questions to the ones missed.

Similar diagnosis were secured for each of the other operations.

The question came up during the employment of the Cleveland Tests as to their correlation with the Courtis Tests. Would a class, for instance, be at any advantage in the showing made by the use of the one rather than the other. For the examination of this point a Grade VI was selected. The

method of procedure was to give the Cleveland Tests in addition, followed by the Courtis Test in addition. The next day a similar course was followed, employing the subtraction series, etc.

The easier sets in each operation as given in the Cleveland Test were omitted as they unduly influenced results. For instance, exercise A contains simple combinations in addition and the class had almost a 100 per cent. record here. For addition, the results of exercises E, J and M were retained; for subtraction, F; for multiplication, G and L; for division, I, K and N. Table XIIA exhibits results. Cl. is the Cleveland results; Co., the Courtis results, while the third column under each operation represents the increase of the Cleveland over the Courtis results.

For addition, it will be noted that there is a very close correlation. The Cleveland Tests favour the class by the small amount of 2.4%. That is the class standing for accuracy by the Cleveland Test is 68.4, whereas the Courtis Test gives 66.

In subtraction, the difference is much greater and consequently reliable data are not furnished the teacher. The reason for this difference will be easily seen upon examination of the types of questions in the two tests. In the Cleveland they are of the type $\begin{array}{r} 854 \\ 286 \\ \hline \end{array}$ whereas the Courtis type is

$$\begin{array}{r} 114857187. \\ 90271797 \\ \hline \end{array}$$

In the case of multiplication the advantage is again in favour of the Cleveland Test by less than 1%. In division, the Courtis Test has the advantage by 1.3%.

The conclusion to be drawn is that with the omissions pointed out a moment ago, the two tests give very close results, except in the case of subtraction.

TABLE XIIA

Pupil	Addition			Subtraction			Multiplication			Division		
	Cl.	Co.	Dif.	Cl.	Co.	Dif.	Cl.	Co.	Dif.	Cl.	Co.	Dif.
1.	64	71	−7	100	86	14	71	67	4	90	100	−10
2.	93	88	5	100	100	0	86	83	3	75	71	4
3.	—	—	—	86	38	18	50	43	7	40	75	−35
4.	91	63	28	100	90	10	100	100	0	73	75	−2
5.	94	67	27	100	88	12	90	90	0	67	86	−19
6.	—	—	—	—	—	—	67	67	0	29	50	−21
7.	—	—	—	100	86	14	50	57	−7	100	100	0
8.	56	67	−11	75	90	−15	67	57	10	—	0	—
9.	77	75	2	100	100	0	66	80	−14	60	80	−20
10.	64	87	−23	—	—	—	50	67	−17	50	75	−25
11.	55	30	25	—	—	—	55	56	−1	71	67	4
12.	70	83	−13	100	43	57	86	80	6	100	100	0
13.	55	86	−31	100	86	14	100	80	20	80	33	47
14.	79	87	−8	63	75	−12	83	100	−17	80	100	−20
15.	41	63	−22	75	30	45	50	86	−36	75	43	32
16.	18	20	−2	38	25	13	17	0	17	25	100	−75
17.	77	63	14	83	50	33	83	60	23	100	25	75
18.	38	50	−12	100	63	37	89	34	55	100	80	20
19.	100	60	10	83	75	8	100	100	0	67	67	0
20.	84	83	1	86	25	61	66	67	−1	100	80	20
21.	75	44	31	60	11	49	56	88	−32	—	—	—

To study the correlation existing between results from the Cleveland Tests and the teacher's estimates, a Grade V of 26 pupils was selected. Table XIII contains the results. Column A is the teacher's estimate in all school subjects; B, estimate in mechanical arithmetic; C, the actual percentages secured from the Cleveland Tests; D, the difference. The closeness of percentages in B and C, except in two or three extreme cases, is quite remarkable and speaks favourably for the reliability of the Cleveland Tests.

One has a suspicion that the teacher's estimate of her pupils' general abilities is based largely upon their abilities in arithmetic.

TABLE XIII

Pupil	A	B	C	D	Pupil	A	B	C	D
A	78	83	75	−8	N	58	52	80	18
B	70	66	69	3	O	72	75	83	8
C	83	79	86	7	P	91	94	97	3
D	80	80	84	4	Q	86	82	91	9
E	73	71	79	8	R	85	85	96	11
F	51	50	58	8	S	70	68	70	2
G	70	65	90	25	T	62	65	69	4
H	68	40	66	22	U	59	40	61	21
I	79	60	69	9	V	50	60	72	12
J	85	90	98	8	W	89	83	95	12
K	68	70	86	12	X	85	82	96	14
L	62	40	81	41	Y	88	92	98	6
M	83	70	84	14	Z	60	58	81	23

It must strike one using any of these tests, where speed is being considered, that the slow writer may be at some disadvantage. This point was not investigated very far, but the variation in speed in writing down figures, was examined in a class of 26. The numbers of figures copied in one minute is given below, as well as the variation from the mean. Column A gives the number of figures copied; B, the variation from the mean.

TABLE XIV

Pupil	A	B	Pupil	A	B
A	85	−33	N	116	−2
B	90	−28	O	120	−2
C	94	−24	P	128	10
D	96	−22	Q	129	11
E	98	−20	R	131	13
F	99	−19	S	132	14
G	103	−15	T	132	14
H	105	−13	U	133	15
I	108	−10	V	137	19
J	108	−10	W	142	24
K	111	−7	X	151	33
L	111	−7	Y	154	36
M	115	−3	Z	155	47

Arithmetic mean 118.

In the Cleveland Tests where some of the intervals given are 30 seconds, considerable of the time must be spent in mere writing down of figures.

While correcting pupils' papers in connection with the Cleveland Tests, it occurred that Set A (simple combinations

in addition) should reveal valuable data as to the more frequent errors and more difficult combinations. An examination of this set will show 65 combinations from 0+0 to 9+9. A Grade II class was selected, which had covered these combinations. While the class contained the usual enrolment, only 17 were present on all of the four trials upon which the tests were given, owing to the epidemic spoken of before.

TABLE XV

	A	B	C		A	B	C
0+0	4	1	1	4+8	4	4	0
0+1	10	4	2	4+9	7	6	1
0+4	6	2	2	5+1	0	0	0
0+5	8	2	2	5+2	1	1	0
0+6	8	2	2	5+5	0	0	0
0+7	7	2	2	5+7	6	3	1
0+8	8	2	2	5+8	9	6	2
1+0	2	2	0	5+9	9	5	2
1+2	1	1	0	6+0	1	0	0
1+3	1	1	0	6+2	0	0	0
1+4	2	2	0	6+3	4	3	1
1+5	0	0	0	6+5	7	4	2
1+6	3	3	0	6+6	2	2	0
2+1	0	0	0	6+9	12	8	3
2+2	0	0	0	7+1	2	2	0
2+4	0	0	0	7+2	3	1	1
2+5	2	2	0	7+4	7	6	2
2+7	4	2	1	7+6	9	5	3
2+8	1	1	0	7+7	4	3	1
2+9	3	3	0	7+8	19	10	4
3+0	1	1	0	7+9	13	8	3
3+1	2	2	0	8+0	1	1	0
3+2	0	0	0	8+1	2	1	1
3+3	0	0	0	8+3	5	2	1
3+4	0	0	0	8+6	11	8	3
3+5	0	0	0	8+9	15	8	4
3+8	6	3	1	9+0	1	1	0
4+0	2	2	0	9+1	1	1	0
4+2	0	0	0	9+3	7	6	1
4+3	1	1	0	9+5	16	10	5
4+5	2	2	0	9+6	15	7	5
4+6	1	1	0	9+8	17	9	5
4+7	3	1	1	9+9	8	5	3

The plan adopted was to give each of the pupils the Cleveland exercise with instructions to do all of Set A and upon completion to stand, when the time of each was recorded. This exercise was given on each of the four successive days. Table XV gives the results. The combinations are arranged in ascending order. Column A gives the total number of

misses for each combination out of 68 attempts (4×17); Column B, the number of children missing the combination once or more; Column C, the number missing twice or more. Table XVI is another arrangement of the same data. Column D, is the percentage of errors to attempts; E, the percentage of pupils missing once or more; F, the percentage of pupils missing twice or more. In each column the combinations are arranged in order of difficulty.

TABLE XVI

Comb.	D	Comb.	D	Comb.	D	Comb.	D
7+8	29	0+6	12	2+7	6	1+0	3
9+8	25	0+5	12	0+0	6	9+1	1
9+5	24	9+3	10	7+2	4	9+0	1
9+6	22	7+4	10	4+7	4	8+0	1
8+9	22	6+5	10	2+9	4	6+0	1
8+6	21	4+9	10	1+6	4	5+2	1
7+9	19	0+7	10	8+1	3	4+6	1
6+9	18	5+7	9	7+1	3	4+3	1
0+1	15	3+8	9	6+6	3	3+0	1
7+6	13	0+4	9	4+5	3	2+8	1
5+9	13	8+3	7	4+0	3	1+3	1
5+8	13	7+7	6	3+1	3	1+2	1
9+9	12	6+3	6	2+5	3		
0+8	12	4+8	6	1+4	3		

Comb.	E	Comb.	E	Comb.	E	Comb.	E
9+5	59	5+9	29	4+0	12	8+0	6
7+8	59	6+5	24	3+1	12	7+2	6
9+8	53	4+8	24	2+7	12	5+2	6
8+9	46	0+1	24	2+5	12	4+7	6
8+6	46	7+7	18	1+4	12	4+6	6
7+9	46	6+3	18	1+0	12	4+3	6
6+9	46	5+7	18	0+8	12	3+0	6
9+6	41	3+8	18	0+7	12	2+8	6
9+3	35	2+9	18	0+6	12	1+3	6
7+4	35	1+6	18	0+5	12	1+2	6
5+8	35	8+3	12	0+4	12	0+0	6
4+9	35	7+1	12	9+1	6		
9+9	29	6+6	12	9+0	6		
7+6	29	4+5	12	8+1	6		

Comb.	F	Comb.	F	Comb.	F	Comb	F
9+8	29	6+9	18	0+5	12	6+3	6
9+6	29	7+4	12	0+4	12	5+7	6
9+5	29	6+5	12	0+1	12	4+9	6
8+9	24	5+9	12	9+3	6	4+7	6
7+8	24	5+8	12	8+3	6	3+8	6
8+6	18	0+8	12	8+1	6	2+7	6
7+9	18	0+7	12	7+7	6	0+0	6
7+6	18	0+6	12	7+2	6		

While the experiment is incomplete from the fact that all the possible combinations are not represented and the number of pupils small, nevertheless it contains possibilities and is suggestive of procedure for a more extended undertaking at a future time.

The matter of difficult combinations is real. In order to supplement the data above, inquiry was made among our teachers-in-training. In one class of 60, all of whom had had teaching experience of one year or more, 19 could recall distinctly, instances of difficulties among their pupils. The others were not sure enough to go on record. Several of the 19 could recall the answers usually met with among their pupils. Typical replies of the 19 are given below.

(1) 7 and 8 are 16. (2) 9 and 8; 9 and 7. (3) 6 and 3; 5 and 4. (4) 9 and 9 are 19. (5) 9 and 7 are 17; 9 and 5. (6) 8 and 6 are 13. (7) 8 and 9; (8) Adding 9 to anything. (9) 9 and 9; 9 and 8; 9 and 7; 9 and 6; 8 and 7; 7 and 6; 5 and 8, etc.

What was even more interesting, 40 of the 60 had difficulties of their own. The following combinations were the troublesome ones:

Combination	No. of persons having difficulty	Combination	No. of persons having difficulty
4 and 3	1	7 and 8	1
4 and 9	1	7 and 9	1
5 and 7	2	8 and 5	4
5 and 8	3	8 and 7	4
5 and 9	1	8 and 9	4
6 and 7	1	9 and 3	3
6 and 8	1	9 and 4	3
6 and 9	2	9 and 5	8
7 and 4	1	9 and 6	10
7 and 5	2	9 and 7	15
7 and 6	3	9 and 8	11

Asked to write down their feelings at the moment when the difficult combination appeared, the following are typical replies: Always had to stop and think; must hesitate for a time; always requires several seconds to decide; hard to find any answer, etc.

The answers to the various combinations varied with different people and even with the same person. Such

answers as the following were very common: 8+7=17; 9+7=17; 8+9=19, etc.

In order to arrive at an understanding of the Woody Scales and to estimate their value to the teacher, or supervisor, as well as their reliability in certain respects, five rooms were used, consisting of some 175 children in Grades IV to VIII. In all, over 700 tests were given and examined in the Woody Scales alone. Table XVII contains the tentative standards proposed by Mr. Woody. Table XVIII represents the achievements attained by the pupils examined.

SERIES A - TABLE XVII

Grade	Addition	Subtraction	Multiplication	Division
IV	6.11	4.22	4.05	3.21
V	6.99	5.47	5.53	4.91
VI	7.95	6.46	6.72	5.87
VII	8.65	7.31	7.26	6.59
VIII	9.01	7.63	7.93	7.16

These standards were obtained from classes during the early part of the term.

SERIES A—TABLE XVIII

Grade	Addition	Subtraction	Multiplication	Division
IV Sr.	6.11	5.85	5.63	4.68
V	7.12	6.64	5.88	4.67
VI	6.99	6.38	6.39	5.11
VII	7.84	7.15	6.75	5.83
VIII	8.49	7.46	7.26	6.71

The standard 6.11 for Grade IV addition means that a problem of this value or difficulty should be solved by one-half the class correctly. Referring to the author's table of values[1] one finds that Problem 20 has a value of 6.10. So that at least half of Grade IV should solve all the addition problems down to and including number 20 (see Appendix).

I have here set out Table XIX containing the performance of a Senior Grade IV of 29 pupils, which will show

[1] Woody: "Measurement of Some Achievements in Arithmetic," page 16

how this works out much better than words. I may say that this class is now being promoted to Grade V.

No. of Problem.... Pupil	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1.	1	1	1	1	1	1	1	–	1	1	–	1	–			
2.	1	1	1	1	1	1	1	1	–	–	1	1				
3.	1	1	1	1	1	1	1	1	1	1	1	–				
4.	1	1	1	1	1	1	1	1	–	–	–	1				
5.	1	1	1	1	1	1	1	1	1	–	1	1	1	1		
6.	1	1	1	1	1	1	1	1	1							
7.	1	1	1	1	1	1	1	1	1	1	1					
8.	1	1	1	1	1	1	1	1	1	1	–	–	1			
9.	1	1	1	–	1	1	1	–	–	1						
10.	1	1	1	1	–	1	–	1	1	–	–	1	–	–	–	1
11.	1	1	1	1	1	–	1	1	1							
12.	1	1	1	1	1	1	–	1	1	1	1	1	1			
13.	1	1	–	1	1	1	1	1	–	–	–	1				
14.	1	1	1	1	1	1	1	1								
15.	1	1	1	1	1	1	1		1	–	1	1				
16.	1	1	1	1	1	-	1	-	1	1	–	–	1			
17.	1	1	1	1	1	1	1	1	–	-	–	–	1			
18.	1	1	1	1	1	1	1	1	1	1	–	1	1	1		
19.	1	1	1	1												
20.	1	1	1	1	1	1	1	1								
21.	–	1	1	1	1	–	-	1	–	1						
22.	1	1	1	1	1	1		1								
23.	1	1	1	1	1	1	-	1	–	–	–	1				
24.	–	1	–		-	-	–	1	–	–	–	1	1			
25.	1	1	1	1	1	1	1	-	1	–	–	1				
26.	1	1	1	1	1	1	–	1	–	-	-	1				
27.	1	1	1	1	1	1	-	1	–	–	–	1				
28.	1	1	1	1	1	-	1	1	–	–	–	1				
29.	1	1	1	–	1	1	–	1	1	–	–	1				
Totals	27	29	27	26	26	21	19	23	14	9	6	16	7	2	-	1

(Practically no errors occurred in the first 10 questions, and these were omitted from the above table. One pupil was successful with number 35, two with 32, while one secured numbers 31 and 33.)

The achievement of this class was a little higher than the Woody norm, due perhaps to the fact that it was a senior grade.

The Woody Scales are useful for diagnosing class and individual errors. Table XX, below, represents the score of a Senior Grade V in addition, Series A, and an analysis of it will indicate how easily the class diagnosis may be made.

TABLE XX

No. of Problem	14	15	16	17	18	19	20	21	22	23	24	25	26
Pupil													
A	1	1	1	1	1	0	0	0	1	1	1	1	0
B	1	1	1	0	1	0	0	0	1	1	1	–	1
C	1	1	1	1	1	1	1	1	1	0	0	0	0
D	1	1	1	1	1	1	1	0	0	0	1	1	0
E	1	1	1	1	1	0	0	0	1	1	0	1	0
F	1	1	1	1	1	1	1	0	0	1	1	1	0
G	1	1	1	1	1	1	0	0	0	1	1	1	0
H	1	1	1	1	1	1	1	1	0	0	1	1	0
I	1	1	1	1	1	1	1	0	0	1	1	1	0
J	1	1	1	1	1	1	1	0	0	1	1	1	0
K	1	1	1	1	1	1	0	1	1	1	1	1	1
L	1	1	0	1	1	1	1	1	0	1	1	1	0
M	1	1	0	1	1	1	0	0	1	1	1	0	0
N	1	1	0	1	1	1	1	1	0	1	1	1	0
O	1	0	1	1	1	0	0	0	1	1	0	0	0
P	1	0	1	0	1	0	0	0	1	0	0	0	–

No. of Problem	27	28	29	30	31	32	33	34	35	36	37	38
Pupil												
A	1	1	0	1	1	1	0	1	0	1	–	–
B	0	0	1	·	1	1	0	–	1	1	–	–
C	0	0	0	0	0	0	0	0	1	1	–	–
D	1	0	–		–	0	0	0	1	1	–	–
E	0	0	1	0	1	0	0	1	1	1	–	–
F	1	0	–	–	·	1	–		1	1	–	–
G	1	1	1	–	1	0	1	1	1	1	–	–
H	1	1	1	0	1	1	1	1	1	–	–	–
I	1	1	0	1	0	0	1	0	1	1	–	–
J	1	0	1	1	0	0	·	·	–	–	–	–
K	1	1	1	1	0	–	–	–	–	–	–	–
L	1	1	1	1	0	1	0	1	1	1	–	–
M	0	0	0	1	1	0	1	1	1	1	–	–
N	0			·	–			1	0	0	–	–
O	1	0	1	1	0	0	0	0	–	–	–	–
P	1	0	0	–	0	0	0	0	0	0	–	–

Here 0 means a question tried and missed; one not attempted. There were no errors in the first 13 questions, and consequently this part of the class score is not indicated.

The table shows that two pupils failed to get number 15. This question consists of column addition of the numbers 100, 33, 45, 201, 46. One answer was 325, the other 435. Three pupils missed 16, a somewhat similar question in addition. Two missed 17, a still more difficult one in addition. No pupil missed 18, a four-column addition of 5 rows. Only one pupil missed two of numbers 15, 16, 17, and doubtless therefore the errors were mere slips. While pupil

P's record is a poor one, still of the 23 questions involving ordinary addition, only two were missed, for inaccuracy. Three others were scored "0" for omission of the $ sign from the answer. A further analysis of the table will indicate the use a teacher should make of the information set out.

As a class, the table shows that number 21 was missed by over half the class. This question required the sum of $8.00, 5.75, 2.33, 4.6, .94, 6.32. The correct answer is $27.50. Practically all the misses were for the omission of the $ sign. From a glance at the way this question is set down, one might be justified in concluding that 27.50 is acceptable as the sum. However, the writer assumed that the Woody norms were calculated on the basis of such an answer being incorrect. As there are a number of such questions throughout the scales, it would be well if the author, in his next edition of his manual, would indicate whether such an answer is acceptable, as it alters the score somewhat appreciably.

Questions 23 to 38 involve considerable skill in decimals, and it is evident that the class as a whole has not had much of this work.

The Woody Scales do not give results that can be calculated in "rate" or per cent.-accuracy, as do the Courtis Tests. These latter measure speed and accuracy as end products, in ordinary addition, subtraction, multiplication and division, while the Scales give the class achievement by indicating the degree of difficulty a question must be to be worked by 50 per cent. of the class. Any comparison of results between the two is therefore a difficult matter.

Added to the difficulty just mentioned is also the fact that the Woody Scales are not limited to "ordinary" addition, subtraction, etc. The Scales, in addition, for instance, contain ordinary addition, addition of dollars and cents, decimals, fractions, denominate numbers; in fact, cover the whole field of addition. The same is true of the other operations.

What may be termed ordinary addition is embraced by questions 1 to 18 and also 22. These 19 questions vary in difficulty from $\frac{2}{3}$ to three-column addition of nine rows. It

would be unsatisfactory to treat these as of equal value and simply find the ratio of the number solved to 19, for the pupils' accuracy. An examination of the method of Mr. Woody in calculating the value of each question convinced the writer that if each of the values were given to the 19 questions, a satisfactory method of estimating the class accuracy, and indeed the individual accuracy too, from the Woody Scales might be evolved. The method in fact would be very similar to the correction of a pupil's answer paper, where the answers had varying values.

With this in mind the 700 odd papers of students in Grades IV to VIII were again examined and marked on the basis of the values given by Woody. In addition questions 1 to 18 and 22 were considered; in subtraction 1 to 19 and 23; in multiplication 1 to 18 and 26; in division 1 to 12 and 14, 16, 17, 18, 19, 21, 23, 24, 25, 33 and 35. Thus in addition, to be 100% correct, the pupil must make a total of 73.54; in subtraction 59.10; in multiplication 66.87; in division 93.85.

Table XXI gives the results by grades with a comparison of performances with the Courtis Tests. It will be seen that Woody Scales, particularly in the first three operations, give a much more favourable standing to the classes.

TABLE XXI

	Addition		Subtraction		Multiplication		Division	
Grade	Co.	W'dy	Co.	W'dy	Co.	W'dy	Co.	W'dy
IV	57	88	76	92	53	80	58	59
V	56	90	66	90	60	77	69	49
VI	58	89	67	90	61	82	69	68
VII	55	88	68	91	69	84	75	71
VIII	61	93	80	97	61	89	66	88

The results in division must not be considered very reliable, particularly for Grades IV to VI. In re-marking the papers, it was difficult to know whether or not certain questions had been attempted by the pupils. Questions not attempted should be discarded in ascertaining accuracy. However, such doubt arose that it was considered best to consider all questions as having been attempted, and this had the effect of lowering the percentage appreciably.

The same lack of uniform progression from grade to

grade as was noticed before is seen here and can be explained largely by the fact that the complexion of the class is constantly changing in a new country.

There is no reason why the plan adopted here could not be extended to the complete scale in each of the operations, and a pupil's ranking secured in percentage. An examination of, say, the addition scale of Woody, convinces one that it contains a good variety of work, and covers pretty well the field of addition; in fact, that it is a fairer test of one's ability in addition than the Courtis Test. The work involved in such valuating, of course, makes the method prohibitive. Could a scale be worked out where the questions increased in difficulty sufficiently to provide simple values such as 2, 4, 5, 6, 8, etc., an excellent test for individuals as well as for class-wholes would be available, which would at the same time be excellent for diagnostic purposes.

Monroe's Reasoning Tests were given to the same grades as were the Courtis Tests, and Woody Scales. Table XXII gives the tentative standards as proposed by Mr. Monroe, Table XXIII the achievements of the classes examined. In reading these one must not overlook the fact that accuracy depends to some extent on the principle involved in the problem. If the principle is wrong the accuracy is marked wrong also, though no actual error in the mechanical work may have occurred.

TABLE XXII

Grade	IV	V	VI	VII	VIII
Correct Principle	9.6	17.0	15.5	20.7	16.8
Correct Answer	5.3	9.7	10.2	14.1	8.4

TABLE XXIII

Grade	IV	V	VI	VII	VIII
Correct Principle	—	14.4	10.9	12.7	16.2
Correct Answer	—	8.3	7.0	7.5	6.7

In order to compare accuracy in mechanical work with the teacher's estimate and ability in reasoning Table XXIV was prepared. This is for a Grade VII class. The teacher's estimates were prepared from records of examinations kept by the teacher, covering several past months.

TABLE XXIV

Pupil	Add.	Subtr.	Mult.	Div.	Aver.	Reas'g	X	Y	Z
A	71	80	83	100	81	89	50	60	63
B	33	70	85	40	61	36	35	65	26
C	75	60	0	50	47	93	25	50	46
D	70	100	75	100	85	68	65	75	43
E	50	75	67	20	56	63	15	50	33
F	17	25	67	20	32	50	20	50	23
G	71	89	80	100	85	74	25	40	46
H	57	78	100	100	83	59	45	65	43
I	75	60	40	33	57	24	40	60	26
J	75	93	100	90	88	56	30	50	46
K	18	36	25	75	37	48	40	65	33
L	14	11	71	75	37	71	40	65	40
M	44	33	67	67	50	59	25	60	53
N	75	44	67	86	67	86	50	60	63
O	50	64	63	75	62	70	50	70	46
P	50	13	13	67	30	38	20	45	26
Q	50	63	50	83	62	88	30	50	50
R	71	83	83	100	81	75	20	40	26
S	85	100	83	100	93	95	35	65	66
T	50	57	100	83	72	72	40	70	33
U	83	86	67	100	83	100	55	75	76
V	70	55	75	83	69	38	40	65	33
W	38	70	88	83	69	73	50	70	53
X	22	89	33	33	50	79	30	45	50

The first five columns are self-explanatory. The sixth, reasoning, contains the percentage of the actual marks secured in principle, based upon the questions actually tried. X is the pupil's record based on monthly examinations covering several months; Y, a similar record for all subjects. Column Z is the ratio in percentage of the correct principle to the total possible, *i.e.*, on the assumption that all problems were tried.

An examination of the table will not indicate any correlation between accuracy in mechanical arithmetic and reasoning. This bears out common experience. A pupil may be good in one and poor in the other, and *vice versa*, or he may be good or bad in both. There is some correlation between reasoning and X, but a closer one between X and Z. The correlation between reasoning and Y (the pupil's record in all school work) is not as close as one should expect, and would suggest that either the tests are not a fair test of reasoning ability or the teacher's monthly examinaions are inadequate in this respect. This has suggested that a very interesting comparison could be made by using the reasoning

tests and some standard intelligence tests, such as Terman's. This did not suggest itself in time for the present thesis.

Table XXV is a Grade VIII record somewhat similar to XXIV. The reasoning column, however, is the ratio of the achievement in principle to the total or complete paper; X the teacher's estimate in general arithmetical ability (mostly in problem work) and Y the estimate for all subjects. The similarity between performances and estimates is more marked than in the previous table. As remarked earlier in this work, however, too much reliance cannot be placed in any existing reasoning tests.

TABLE XXV

Pupil	Add.	Subtr.	Mult.	Div.	Aver.	Reas'g	X	Y
A	55	92	90	88	75	72	85	85
B	91	78	100	100	92	91	80	80
C	56	86	71	100	79	54	60	80
D	57	89	75	83	77	34	60	75
E	88	88	89	57	81	59	65	75
F	75	57	86	100	79	50	65	68
G	50	100	100	100	89	10	50	60
H	46	85	82	85	74	34	45	50
I	56	78	60	67	68	72	60	72
J	75	75	89	100	85	50	70	65
K	89	78	17	75	68	47	35	60
L	91	82	89	67	83	84	65	65
M	63	55	89	88	72	—	55	60
N	14	67	67	50	50	—	30	40
O	75	82	89	92	83	54	50	50
P	56	100	78	88	81	91	50	65
Q	75	50	43	75	61	41	30	50
R	83	86	100	100	92	22	50	60
S	50	67	91	90	78	34	50	60
T	71	90	57	100	81	56	70	75
U	73	88	73	89	81	91	75	78
V	50	57	83	100	74	44	45	60
W	67	57	50	75	71	59	45	50
X	25	90	44	100	68	62	60	55
Y	55	67	100	92	77	54	85	75
Z	67	100	80	67	80	28	60	68
AA	0	57	50	20	32	50	35	50
AB	11	86	78	70	60	60	75	65
AC	100	100	82	86	92	92	70	80
AD	50	80	75	100	74	—	30	40
AE	78	78	57	75	72	7	30	50
AF	50	63	50	83	62	41	30	50
AG	33	67	57	86	62	34	55	65
AH	25	50	25	0	29	31	45	60
AI	50	56	57	83	61	41	55	65
AJ	17	78	83	100	72	47	75	74
AK	67	89	80	100	85	47	80	72
AL	75	100	90	88	89	47	75	80

In conclusion, one should say it would be a mistake in the present stage of the development of these objective measurements, to make them the sole test of progress or the character of the teacher's work. Comparisons are always fraught with dangers. It appears, also, that in comparing abilities too much may be made of the time element. Students, we know, work at different rates, and the same student had various rates from day to day, depending upon many factors. A partial solution is of course to take an average of several performances.

A writer in Teachers College Record[1] commenting upon secondary school tests, raises, to my mind, a very pertinent point, which is applicable in the field of elementary arithmetic too. We have not full information yet as to the complex mental traits underlying the learning of any subject. Is it not possible that in measuring the more obvious and consequently more superficial elements, we are missing the really subtle and hidden qualities and possibly the most important.

The same writer raises a doubt about the "times" judgment in mental traits. I shall quote in part. ". . . . We are dealing here with the increment of ability which is exceedingly difficult, if not impossible, to evaluate. We may illustrate the point concretely as follows: It is reported that Charlie Chaplin has signed a contract calling for a million dollars per year salary, whereas, North, who imitates him, gets but $1,000 per year. Shall we say that Chaplin is 1,000 times as efficient as North? The person who does not attend the movies regularly probably is unable to distinguish the two characters. Chaplin is *just a little* better than his imitator, but it is this small increment which is of greatest value because, as Thorndike points out, '. . . they occur seldom, become famous and are given large financial returns.' Thorndike shows that though we may judge differences in intellectual products near the middle of the human range by their intrinsic quantity and quality we shift our basis of judgment as the limit is approached. Thorndike is skeptical about the logic of the 'times' judgment in

[1] November, 1917.

mental traits, and careful in the use of the phrase 'times as efficient.' The point is more easily illustrated in the discussion of improvement, as it is easier to see that an improvement from just being able to solve four problems per minute to just being able to solve five per minute is an altogether different improvement from ten to eleven."

These doubts that have been raised do not detract from the importance of the measurements but suggest that they may have not yet reached a state of perfection. In any event there will always remain some elements, and very desirable ones at that, of mental ability which defy objective measurement. And likewise certain aspects of the teacher's instruction will never show tangible results. So long as this remains true the poor pedagogue can be deluded into the belief that his reward is not of this earth, but that he is laying up treasures where neither moth nor rust doth corrupt and where thieves do not break through and steal.

It should also be noted that the field of what we might call the "technique" of arithmetic has barely been entered. Several experiments have been lightly described in the preceding pages. Many matters await further verification and many others scientific study along the lines of experimental investigation particularly. There is usually a protest when any proposed change is mooted and a generation or more may pass before the improvement is finally adopted and we wonder how our "fathers" could have been so slow to recognize its merits. The history of the development of the subject is replete with such instances. The introduction of our present system of numerals, the establishment of our symbols of operations, decimal fractions, etc., are a few cases in point. Future generations will no doubt regard us as we do the past generations. For instance, why cannot we adopt the metric system? The past has seen several improvements in multiplication and division, why are we so slow in adopting the practice of placing the quotient above the dividend in long division, instead of to the right, and thus letting the decimal point take care of itself? Why do we persist in solving "Goods sold for $396 at a loss of 10%; find the cost," in this way: 90% of cost of goods, etc., when we might just as well say $9x = $396, etc. Each genera-

tion is conservative and seems satisfied to let well enough alone.

A few of the possible lines of investigation along the line of technique will be here suggested. Several have already been hinted at, for example, the best arrangement of number pictures, value of number forms, use of play and recreation, etc.

We have already agreed that the number range in Grade I should be broadened. Within this large range should group counting, such as by 2's, 3's, 5's, etc., be made much of as "counting," or is it merely addition and should wait until we are dealing with addition as an operation? Should not counting backward be made an important practice, as Browne suggests,[1] so as to strengthen the subtractive associations and thus result in economy in the operation of subtraction? Within this broader first year range, say 1 to 100, is it advisable when teaching, say, 4 and 9 are 13, to also teach the extensions, as 14 and 9 are 23, 24 and 9 are 33, etc.?

It would seem that further investigation along the line suggested[2] in order to establish what numbers are favourites, what combinations are in general difficult, etc., would be worth while so that the teacher would have some guide as to the order of the numbers to be taught as well as what particular combinations should get extra drill. This would eliminate the wasteful overlearning of facts which come easily to the child. Are there any basic combinations which should be taught first? In teaching combinations we ordinarily teach $3+4=7$ and latterly, $\frac{3}{4}$. Is this economical as well as psychological? When teaching $3+4=7$, should we immediately teach $4+3=7$?

Under present practice the child eventually learns the multiplication table in order, thus: $4\times5=20$, $4\times6=24$, $4\times7=28$, etc. Is there a better order as Browne would seem to hint? Is his contention sound that by early use of the commutative law, as $3\times9=9\times3$, the tables could be cut in two?

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[1] "American Journal of Psychology," 1906, p. 30.
[2] *Supra*, p. 140

Many pupils know their number facts and tables but are not accurate in the fundamental operations. This raises an important question as to the relation between such knowledge and accuracy.

These are but a few of the possible lines of inquiry having in mind improved technique. Possibly no subject presents a greater field for experimentation than arithmetic, and yet no subject has been so little investigated. It is to be hoped that this reproach will soon be removed.

APPENDIX A.

BIBLIOGRAPHY.

The following are among the books consulted:

Bagley	Class Room Management
	Educative Process
	Educational Values
Ball	A Short History of Mathematics
	Mathematical Recreations and Essays
Cajori	History of Elementary Mathematics
Colvin	The Learning Process
Colvin and Bagley	Human Behaviour
Conant	The Number Concept
Courtis	Standard Research Tests
Cyclopædia of Education	
De Garmo	Essentials of Method
	Interest and Education
Dewey	How We Think
	Interest and Effort in Education
	The School and Society
	Schools of Tomorrow
Earhart	Types of Teaching
Encyclopædia of Education	Articles on Arithmetic, etc.
Francis Parker	Year Books
Froebel	Tr. Fletcher and Welton
Galton	Inquiries into the Human Faculties
Craven	History of Education
Grube	Tr. Seeley
Hall	Adolescence
Horne	Philosophy of Education
Howell	Pedagogy of Arithmetic
James	Psychology
Johnson	Education by Plays and Games
Jones	Education as Growth

Judd	Psychology
	Psychology of High School Subjects
Kendall and Mirick	How to Teach the Fundamental Subjects
Kennedy	Fundamental Methods
McLellan and Dewey	Psychology of Number
McMurry	Special Method in Arithmetic
	Elements of General Method
Miller	The Psychology of Thinking
Monroe, De Voss and Kelly	Educational Tests and Measurements
Munsterberg	Psychology and the Teacher
Myers	Introduction to Experimental Psychology

Ontario Teachers' Manual in Arithmetic

O'Shea	Education as Adjustment
	Dynamic Factors in Education
Pestalozzi	How Gertrude Teaches Her Children (trans. Holland and Turner)
	Educational Writings (trans. Green)
Parker	History of Education

Reports of Committee of Ten and Fifteen

Rogers	Tests of Mathematical Ability and Their Prognostic Value
Rose and Lang	Groundwork of Number
Rousseau	Davidson
Rusk	Introduction to Experimental Education
Sandiford	Mental and Physical Life of School Children
Scott	Social Education
Sleight	Educational Values and Methods
Smith	The Teaching of Elementary Mathematics
Speer	Arithmetic
Starch	Experiments in Educational Psychology

Stone	Arithmetical Abilities and Some Factors Determining Them
Stone and Mills	Arithmetics
Suzzallo	The Teaching of Primary Arithmetic
Thorndike	Educational Psychology (Briefer)
Watson	Outlines of Philosophy
Whipple	Manual of Mental and Physical Tests
White	Scrap Book of Elementary Mathematics
Wilson and Wilson	Motivation of School Work
Woody	Measurements of Some Achievements in Arithmetic
Young	Teaching of Mathematics

Access was had to numerous journals and magazines, such as Educational Foundations, Educational Review, Education, Elementary School Teacher, Inspectors' Reports for Province of Saskatchewan, American Journal of Psychology, Journal of Educational Psychology, Teachers College Record, Pedagogical Seminary, Year Books of National Society for Study of Education, various Courses of Study, arithmetic texts, etc.

APPENDIX B.

Extracts from typical Courses of Study for the work of the first year,

A.—Saskatchewan.

Numbers 1 to 10. Easy numerical relations in grouping and separating objects; combinations and separations, including such mental work as:

$4+5=$
$10-3=$
$4\times2=$
A half of 6 is?
How many "fours" in 8?

Use of arithmetical signs necessary in expressing these facts.

Writing the numbers 1 to 10 in figures and in Roman numerals.

Common units of measurement; inch, pint, quart, day, week, five cent and ten cent coins.

Oral solutions of easy problems based upon the above.

B.—Horace Mann Elementary School.

Introductory note: The First Grade aims primarily to develop the number concept by taking advantage of the many opportunities for teaching number offered by the various class room activities. Definite number work within the limits of 1 to 100 is also given.

FIRST GRADE.

The chief aim in this grade is to develop the number concept by using the children's number experiences and needs for counting, comparing and measuring.

Counting—Groups of objects; children marching by twos; children seated five in a row. Counting pupils in the class and enough paper for them.

Reading Numbers—Pages of books; calendars; street signs; house numbers; Roman numerals to XII in the study of the clock face.

Writing Numbers—Numbers to 100. Attention is given to the correct formation of figures, especially 4, 8, 7, 9.

Measures—Day, week, month, year; cent, nickel, dime, dollar. Measuring materials used in industrial arts, involving the use of inch, foot, yard, pound, dozen, half dozen; the fraction one-half inch; *e.g.*, measuring the materials for looms and counting the number of nails needed for them.

Approximate and Exact Comparison of Size and Number.—Approximate comparison, *e.g.*, Laura is taller than Milton; Margery's score is larger than Herman's. Exact comparison, *e.g.*, this block is twice as large as that block.

Addition and Subtraction—Numbers suggested by games and other interests of children.

Problems—Problems suggested by children's interests and experiences. Children are encouraged to suggest problems.

Games—Bean Bag; Marbles; Odd or Even; Ring Toss; Hull Gull; Dominoes; Guessing Game; Tag.

C—McMurry's Suggested Course.

FIRST GRADE—INCIDENTAL NUMBER WORK.

In our course of study we have made no provision for regular number work in the first year. Our presumption is that it is better for children of this age to gather number experiences incidentally from home and school employments. The regular and systematic drill on number combinations in the first year seems to us premature, and the time thus spent can be better employed in widening a child's experiences in nature and in human affairs. With this accumulation of experiences and with greater maturity, children may grapple with number more effectively the second year.

The recent widening of the activities of primary children into nature study, school games, literature, drawing, and constructive arts, gives a much richer number experience in the first year.

By incidental number work it is meant that where quantitative relations are present, enough attention shall be given to them to make the ideas clear. This is desirable even from the standpoint of nature study, of stories, and of constructive exercises, etc. But this can be easily overdone. It is not our aim to make construction or weather study merely a vehicle for bringing out number relations. The idea is to let number ideas grow naturally, and not to force them.

The following outline indicates a few of the instances where number appears and can receive this incidental attention:

1. The number of children in the school and in different classes. The relative number of boys and girls. The school enrolment and number in attendance. Absences and tardiness.

2. Distributing and collecting materials for class use, as pencils, books, pens, blotters. A monitor for each row can report the number needed for use in his row.

APPENDIX C.

NUMBER GAMES.

These are arranged alphabetically, not in order of use in the class room.

Around the Circle—The digits are arranged in the form of a clock face upon the blackboard. Any number of these digits may be used to fit the grade of work. A digit is then placed in the centre and the numbers are multiplied by it as rapidly as possible. If a pointer can be made to swing on a pivot, the game may be varied by taking the numbers on which the pointer rests. The game can be varied so as to relate to addition and subtraction.

Backgammon—This well-known game involves both addition and subtraction.

Baseball Percentages—The averages of the team in the school or in the national leagues are always interesting to the boys. The principle may also be applied to the players' averages.

Bean Bag—Each player throws two bean bags at a board in which there are holes of different sizes, the count for each being different. A better way is to play a group against another group having scorers for each side. By making penalty holes, subtraction may also be introduced into the game.

Bird Catcher—A variation of this well known game is to have the children sit in a circle, each taking a number. The pupil in the centre gives easy examples. When a result is the number of any pupil, he holds up his hands. When a number already agreed upon is the result, all hold up their hands.

Blackboard Relay—The class is divided into sections having equal numbers. The leader is given a piece of chalk. At the word of command he goes to the board and does the example assigned to him, returning to the next child when he has finished, and so on to the last. Correctness, neatness, and good order may be made factors in winning. Any errors in the work may be corrected by a succeeding pupil, this keeping in mind not merely the work itself but also the formation of the desired habit.

APPENDIX D.

SERIES A

ADDITION SCALE

By Clifford Woody

City. . . . County. . . School . . . .

Date. . . Name . . . . . .

When is your next birthday? . . . . How old will you be? . .

Are you a boy or a girl? ..... In what grade are you? ....

Teacher's name ..... .

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
2	2	17	53	72	60	3+1=	2+5+1=	20
3	4	2	45	26	27			10
—	3	—	—	—	—			2
	—							30
								25
								—

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
21	32	43	23	25+42=	100	9	199	2563
33	59	1	25		33	24	194	1387
35	17	2	16		45	12	295	4954
—	—	13	—		201	15	156	2065
		—			46	19	——	——
					——	—		

(19)	(20)	(21)	(22)	(23)	(24)	(25)
$ .75	$12.50	$8.00	547	$\frac{1}{3}+\frac{1}{3}=$	4.0125	$\frac{3}{8}+\frac{5}{8}+\frac{7}{8}+\frac{1}{8}=$
1.25	16.75	5.75	197		1.5907	
.49	15.75	2.33	685		4.10	
———	———	4.16	678		8.673	
		.94	456		———	
		6.32	393			
		———	525			
			240			
			152			
			——			

(26)	(27)	(28)	(29)	(30)	(31)	(32)
$12\frac{1}{2}$	$\frac{1}{8}+\frac{1}{4}+\frac{1}{2}=$	$\frac{3}{4}+\frac{1}{4}=$	$4\frac{3}{4}$	$2\frac{1}{2}$	113.46	$\frac{3}{4}+\frac{1}{2}+\frac{1}{4}=$
$62\frac{1}{2}$			$2\frac{1}{4}$	$6\frac{3}{8}$	49.6097	
$12\frac{1}{2}$			$5\frac{1}{4}$	$3\frac{3}{4}$	19.9	
$37\frac{1}{2}$			—	—	9.87	
—					.0086	
					18.253	
					6.04	
					———	

(33)	(34)	(35)	(36)	(37)
.49	$\frac{1}{6}+\frac{3}{8}=$	2 ft. 6 in.	2 yr. 5 mo.	$16\frac{1}{3}$
.28		3 ft. 5 in.	3 yr. 6 mo.	$12\frac{1}{8}$
.63		4 ft. 9 in.	4 yr. 9 mo.	$21\frac{1}{2}$
.95		————	5 yr. 2 mo.	$32\frac{3}{4}$
1.69			6 yr. 7 mo.	——
.22			————	
.33				
.36				
1.01				
.56				
.88		(38)		
.75	$25.091+100.4+25+98.28+19.3614=$			
.56				
1.10				
.18				
.56				
——				

SERIES A

SUBTRACTION SCALE

By Clifford Woody

City..... ....County..... .... School ....

Date..... ..... School .... Date.. . . ..

When is your next birthday?.. ... .How old will you be?. ....

Are you a boy or a girl?..... . ...In what grade are you? ... ...

Teacher's name . .. .

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
8	6	2	9	4	11	13	59	78	7 − 4 =	76
5	0	1	3	4	7	8	12	37		60
—	—	—	—	—	—	—	—	—		—

(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
27	16	50	21	270	393	1000	567482	$2\frac{3}{4} - 1 =$
3	9	25	9	190	178	537	106493	
—	—	—	—	—	—	—	—	

(21)	(22)	(23)	(24)	(25)	(26)
10.00	$3\frac{1}{2} - \frac{1}{2} =$	80836465	$8\frac{7}{8}$	27	4 yds. 1 ft. 6 in.
3.49		49178036	$5\frac{3}{4}$	$12\frac{5}{8}$	2 yds. 2 ft. 3 in.
—		—	—	—	—

(27)	(28)	(29)	(30)
5 yds. 1 ft. 4 in.	10 − 6.25 =	$75\frac{3}{4}$	9.8063 − 9.019 =
2 yds. 2 ft. 8 in.		$52\frac{1}{4}$	
—		—	

(31)	(32)		(33)	(34)	(35)
7.3 − 3.00081 =	1912	6 mo. 8 da.	$\frac{5}{12}$ $\frac{2}{10}$ =	$6\frac{1}{8}$	$3\frac{7}{8} - 1\frac{2}{3} =$
	1910	7 mo. 15 da.		$2\frac{7}{8}$	
		—		—	

SERIES A

MULTIPLICATION SCALE

By Clifford Woody

City County School

Date Name

When is your next birthday? How old will you be?

Are you a boy or a girl? In what grade are you?

Teacher's name

(1) $3 \times 7 =$

(2) $5 \times 1 =$

(3) $2 \times 3 =$

(4) $4 \times 8 =$

(5) 23 × 3 =

(6) 310 × 4 =

(7) $7 \times 9 =$

(8) 50 × 3 =

(9) 254 × 6 =

(10) 623 × 7 =

(11) 1036 × 8 =

(12) 5096 × 6 =

(13) 8754 × 8 =

(14) 165 × 40 =

(15) 235 × 25 =

(16) 7898 × 9 =

(17) 145 × 206 =

(18) 24 × 234 =

(19) 9.6 × 4 =

(20) 287 × .05 =

(21) 24 × $2\frac{1}{2}$ =

(22) $8 \times 5\frac{3}{4} =$

(23) $1\frac{1}{4} \times 8 =$

(24) $16 \times 2\frac{5}{8} =$

(25) $\frac{7}{8} \times \frac{3}{4} =$

(26) 9742 × 59 =

(27) 6.25 × 3.2 =

(28) .0123 × 9.8 =

(29) $\frac{1}{5} \times 2 =$

(30) 2.49 × .36 =

(31) 12 15 − × − = 25 32

(32) 6 dollars 49 cents × 8 =

(33) $2\frac{1}{2} \times 3\frac{1}{2} =$

(34) $\frac{1}{2} \times \frac{1}{2} =$

(35) $987\frac{3}{4} \times 25 =$

(36) 3 ft. 5 in. × 5 =

(37) $2\frac{1}{4} \times 4\frac{1}{2} \times 1\frac{1}{2} =$

(38) $.0963\frac{1}{8} \times .084 =$

(39) 8 ft. $9\frac{1}{2}$ in. × 9 =

SERIES A

DIVISION SCALE

By Clifford Woody

City..... County..... School.....

Date..... Name.....

When is your next birthday?..... How old will you be?.....

Are you a boy or a girl? ..... In what grade are you?.....

Teacher's name.....

(1) $3\overline{)6}$ (2) $9\overline{)27}$ (3) $4\overline{)28}$ (4) $1\overline{)5}$ (5) $9\overline{)36}$ (6) $3\overline{)39}$

(7) $4 \div 2 =$ (8) $9\overline{)0}$ (9) $1\overline{)1}$ (10) $6 \times \ldots = 30$ (11) $2\overline{)13}$ (12) $2 \div 2 =$

(13) $4\overline{)24 \text{ lbs. } 8 \text{ oz.}}$ (14) $8\overline{)5856}$ (15) $\frac{1}{4}$ of $128 =$ (16) $68\overline{)2108}$ (17) $50 \div 7 =$

(18) $13\overline{)65065}$ (19) $248 \div 7 =$ (20) $2.1\overline{)25.2}$ (21) $25\overline{)9750}$ (22) $2\overline{)13.50}$

(23) $23\overline{)469}$ (24) $75\overline{)2250300}$ (25) $2400\overline{)504000}$ (26) $12\overline{)2.76}$

(27) $\frac{7}{8}$ of $624 =$ (28) $.003\overline{).0936}$ (29) $3\frac{1}{2} \div 9 =$ (30) $\frac{3}{4} \div 5 =$

(31) $\frac{5}{4} \div \frac{3}{5} =$ (32) $9\frac{5}{8} \div 3\frac{3}{4} =$ (33) $52\overline{)3756}$

(34) $\text{02.50} \div 1\frac{1}{4} =$ (35) $531\overline{)37722}$ (36) $9\overline{)69 \text{ lbs. } 9 \text{ oz.}}$

APPENDIX E.

SEPARATE TYPES OF EXAMPLES IN THE OPERATIONS WITH INTEGERS.[1]

Addition—Addition combinations; single column addition of three figures each; "bridging the tens" as 38—7; column addition, seven figures; carrying; column addition with increased attention span, thirteen figures in the column; addition of numbers of different lengths.

Subtraction—Subtraction combinations; subtraction of 9 or less from a number of two digits, both with and without simple "borrowing"; subtraction involving borrowing.

Multiplication—Multiplication combinations; multiplicand two digits, multiplier one digit, and no carrying; same as the preceding but with carrying; long multiplication, without carrying; zero difficulties; long multiplication, with carrying.

Division—Division combinations, simple division, no carrying, same as the preceding but with carrying; long division, no carrying; zero difficulties, without carrying, long division, with carrying, "first case," the first figure of the divisor is the trial divisor and the trial quotient is the true quotient; "second case where the trial divisor is the one larger than the first figure of the dividend, but the trial quotient is the true quotient;" "third case, where the first figure of the divisor is the trial divisor, but the true quotient is one smaller than the trial quotient;" "fourth case, where the first figure of the divisor must be increased by one to get the true quotient."

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[1] Adapted by Monroe, De Voss and Kelly from Courtis: "Manual for Courtis Standard Practice Tests."

APPENDIX F.

Brief Summary of Reasons, from Professor Smith's Address to the Faculty.[1]

(1) Mathematics is one of the small group of subjects—like history, reading, geography—that are linked up with a larger number of the branches of human knowledge. The author draws a very interesting picture of what would happen in the event of every mathematical book, sheet, etc., and every machine for computing and recording numbers were suddenly wiped off the face of the earth.[2]

(2) Mathematics has a high value as a mental discipline.

(3) Mathematics has a poetical side as well as a practical. "The beauty of symmetry—where else is it found as completely as in mathematics, and where does rhythm play as great a part as here? Why was music looked upon until recently as a part of mathematics if not for the common elements of the two? Why does nature so often arrange the leaves of plants in accordance with a Fibonacci series, and why does the snow crystal recognize the poetry of the complex sixth roots of unity? You do not recall being taught all this? Then the argument is against the pedagogue, still so often a slave as in ancient times; it is not against the poetry of mathematics."

(4) Mathematics is "one of the eternal verities. . . . Did you ever think how you might proceed to make an attempt to communicate with Mars by signals? . . . No, it would be none of these; the most hopeful symbol we could give to attract the attention of a world much older than our own, and probably more refined, would be the figure of the theorem of Pythagoras, the square on the three sides of a right angled triangle. . . . You cannot think of a

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[1] Smith: Address before Faculty, Teachers College, Record for 1917.

[2] Another picture is drawn by a writer in Educational Review, January, 1917, in the article, "Mathematics: A Great Inheritance."

better symbol; and the reason is that here is one of the verities of the universe."

(5) Mathematics makes one conscious of his position in the universe around him.

(6) The study of mathematics gives humanity a religious sense that cannot be fully developed without it. "In the history of the world, mathematics has its genesis in the yearning of the human soul to solve the mystery of the universe in which it is a mere atom. . . . "

(7) The history of mathematics is the history of the race.

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